Nonlinear Fourier Analysis Of Ocean Waves

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  - Solomon Yim
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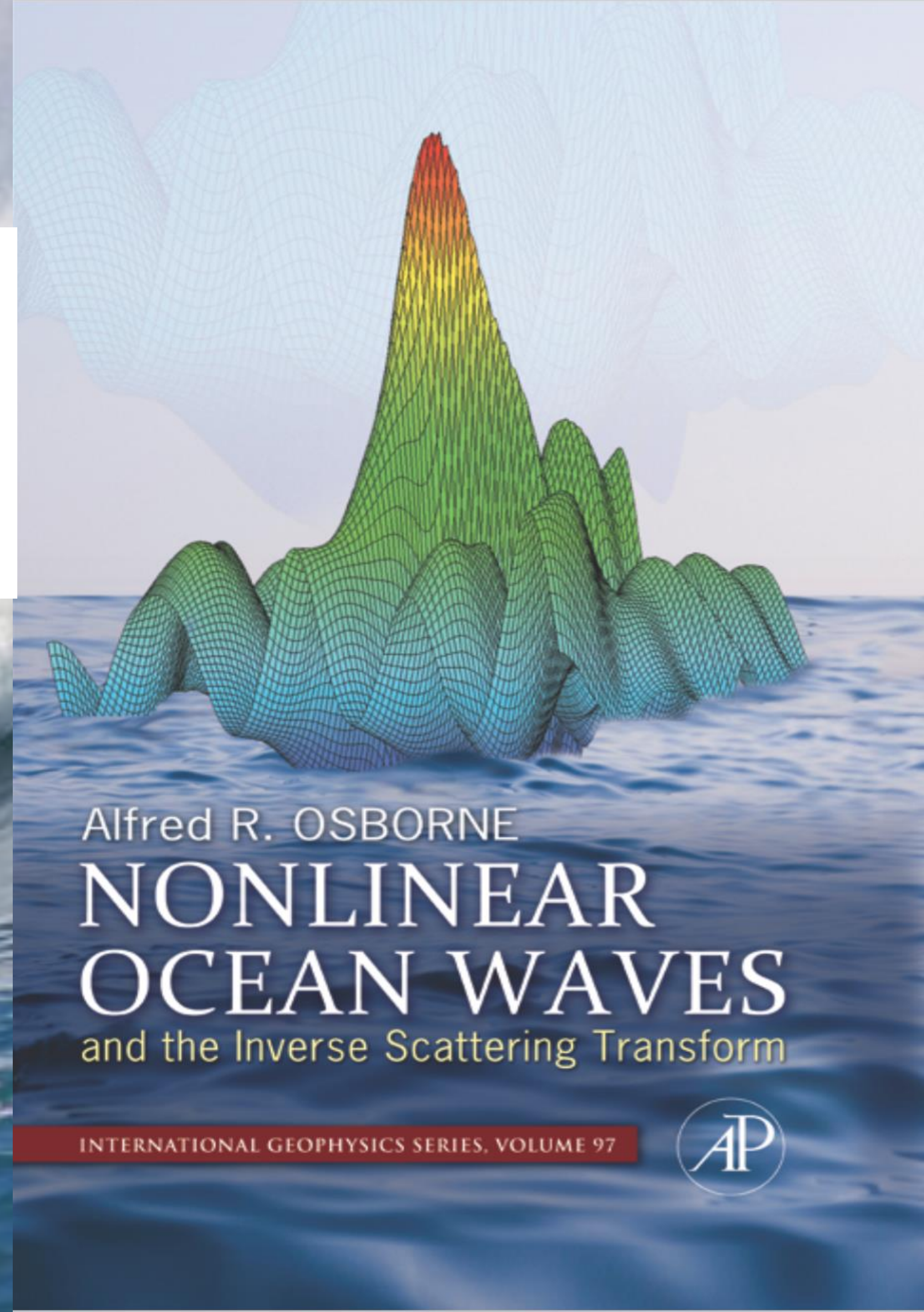
  - Uggo de Pinho
- ***Internal Waves***
  - Gus Jeans
  - Chris Jackson
- ***Brain Waves***
  - Francesco Osborne
- ***Management***
  - Patrizia Osborne
  - Dakota Osborne

Nonlinear Ocean Waves and
The Inverse Scattering Transform
Alfred R. Osborne
Academic Press, Boston, 2010
950 pages.



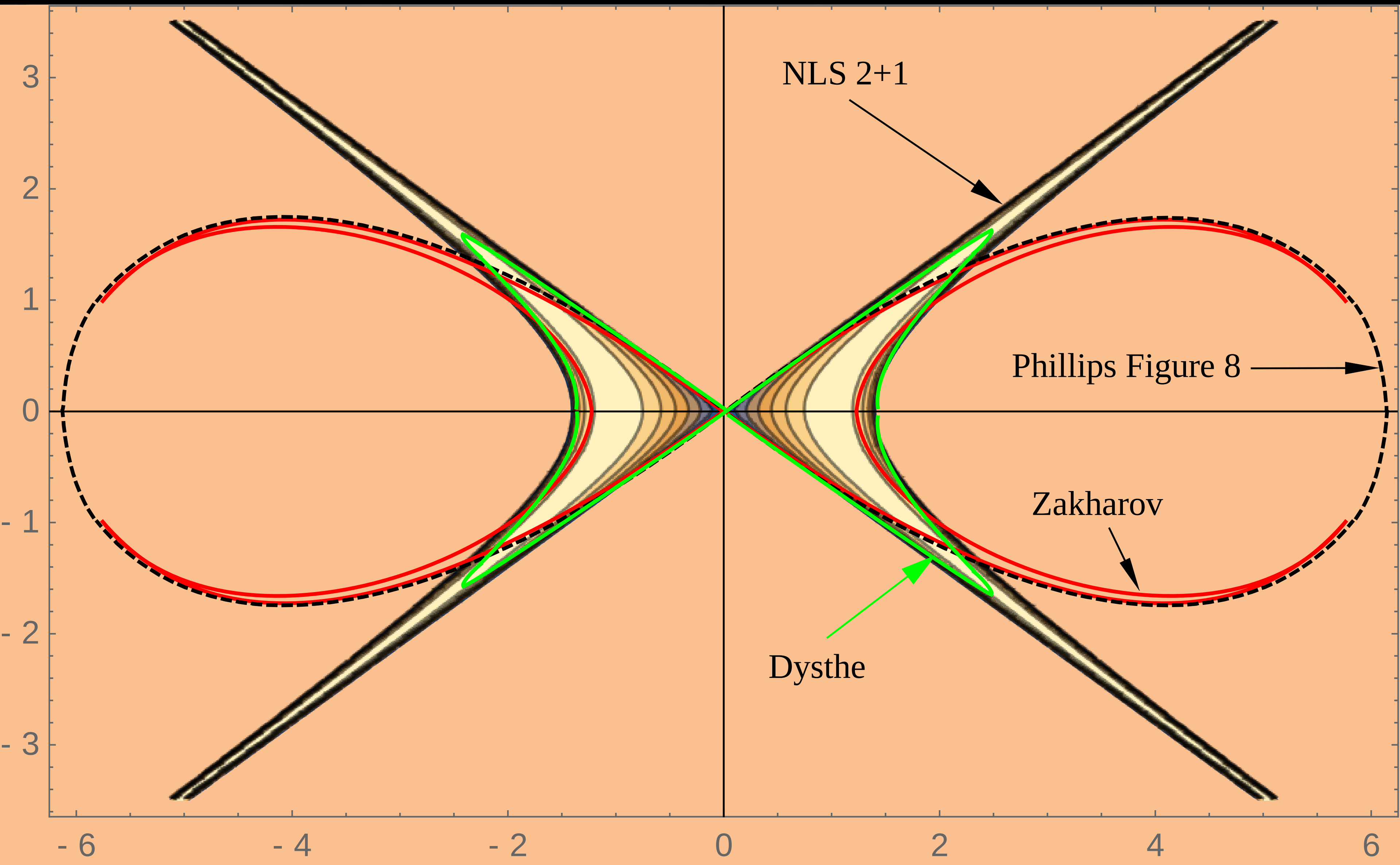
Recent & Relevant Work

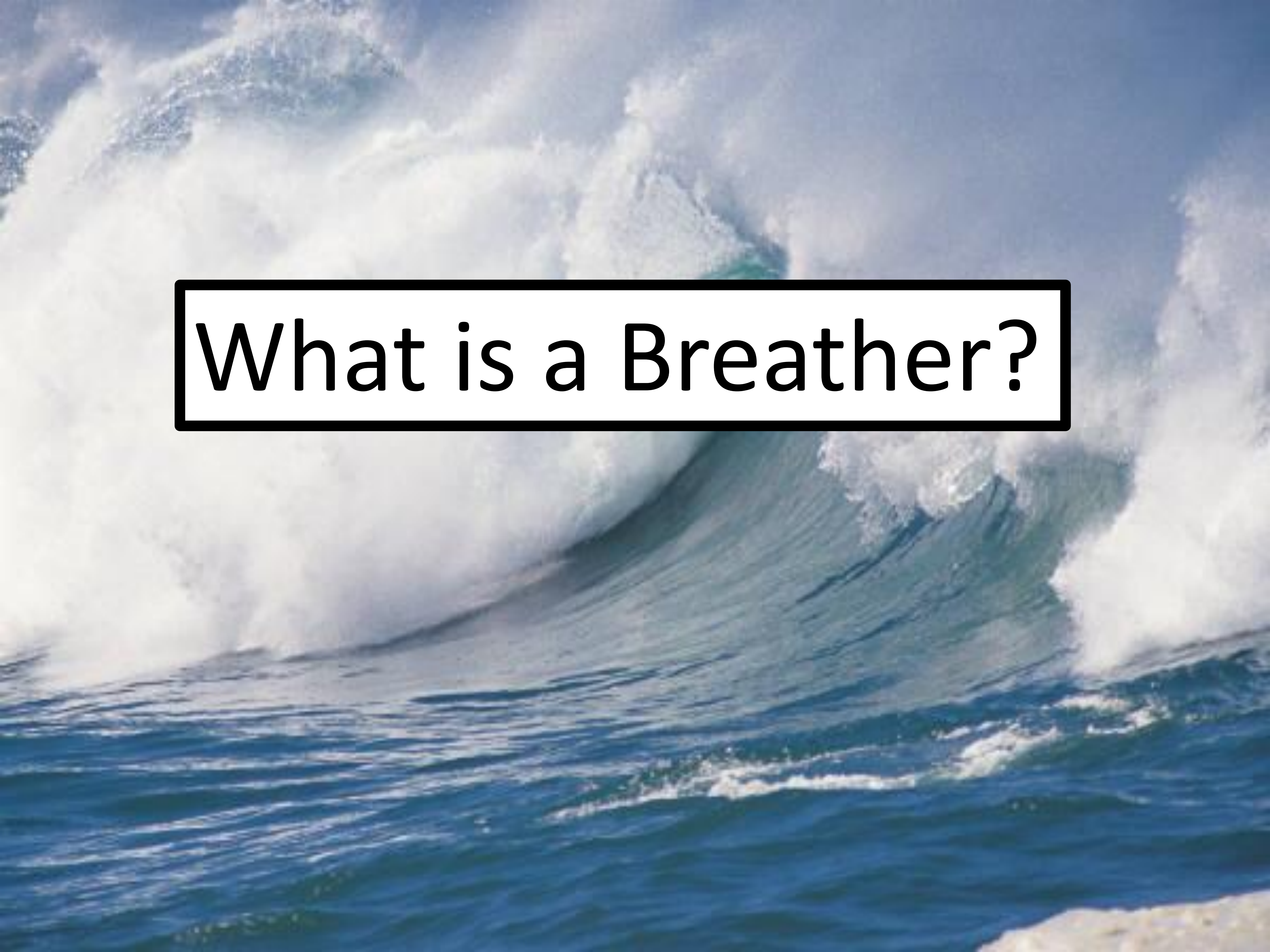
- Osborne, A. R., The simulation and measurement of random ocean wave statistics. In: Osborne A. R., Malanotte-Rizzoli P (eds.) Topics in ocean physics. North- Holland, Amsterdam, 1982.
- Osborne, A. R., The behavior of solitons in random-function solutions of the periodic Korteweg-deVries equation. Phys. Rev. Lett. 71(19):31153118, 1993.
- Osborne, A. R., Nonlinear ocean waves and the inverse scattering transform. Academic Press, Boston, 2010.
- Costa, A., A. R. Osborne, D. T. Resio, S. Alessio, E. Chrivì, E. Saggese, K. Bellomo, and C. E. Long, Soliton Turbulence in Shallow Water Ocean Surface Waves, Phys. Rev. Lett. 113, 108501, 2014.
- Osborne A. R., Nonlinear Fourier methods for ocean waves. Procedia IUTAM, IUTAM Symposium Wind Waves, 48, London, 2018.
- Osborne, A. R., Resio, D. T., Costa, A., Ponce de León, S.; Chirivì, E. Highly Nonlinear Wind Waves in Currituck Sound: Dense Breather Turbulence in Random Ocean Waves, Ocean Dynamics, https://doi.org/10.1007/s10236-018-1232-y, 2018.
- Osborne, A. R., Breather Turbulence: Exact Spectral and Stochastic Solutions of the Nonlinear Schrödinger Equation, Fluids, 4, 2019.

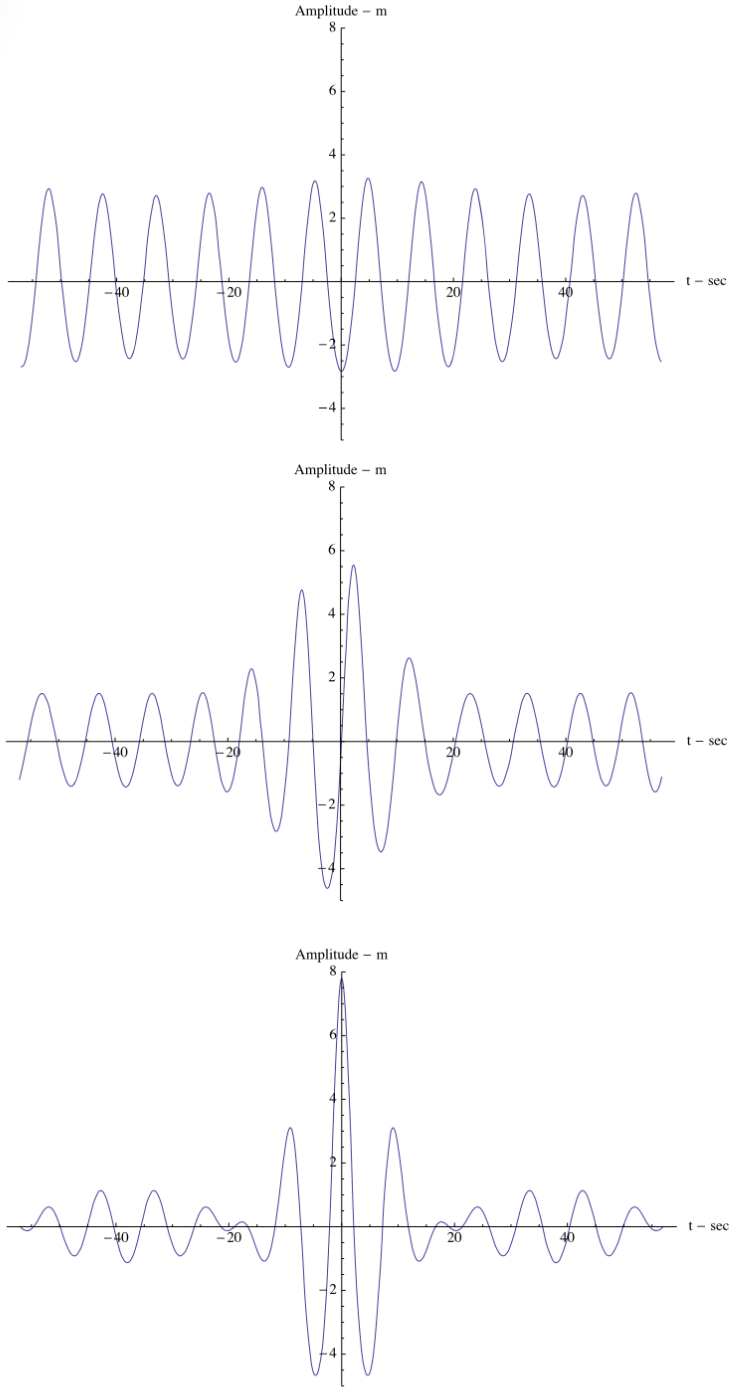
Ocean Surface Waves

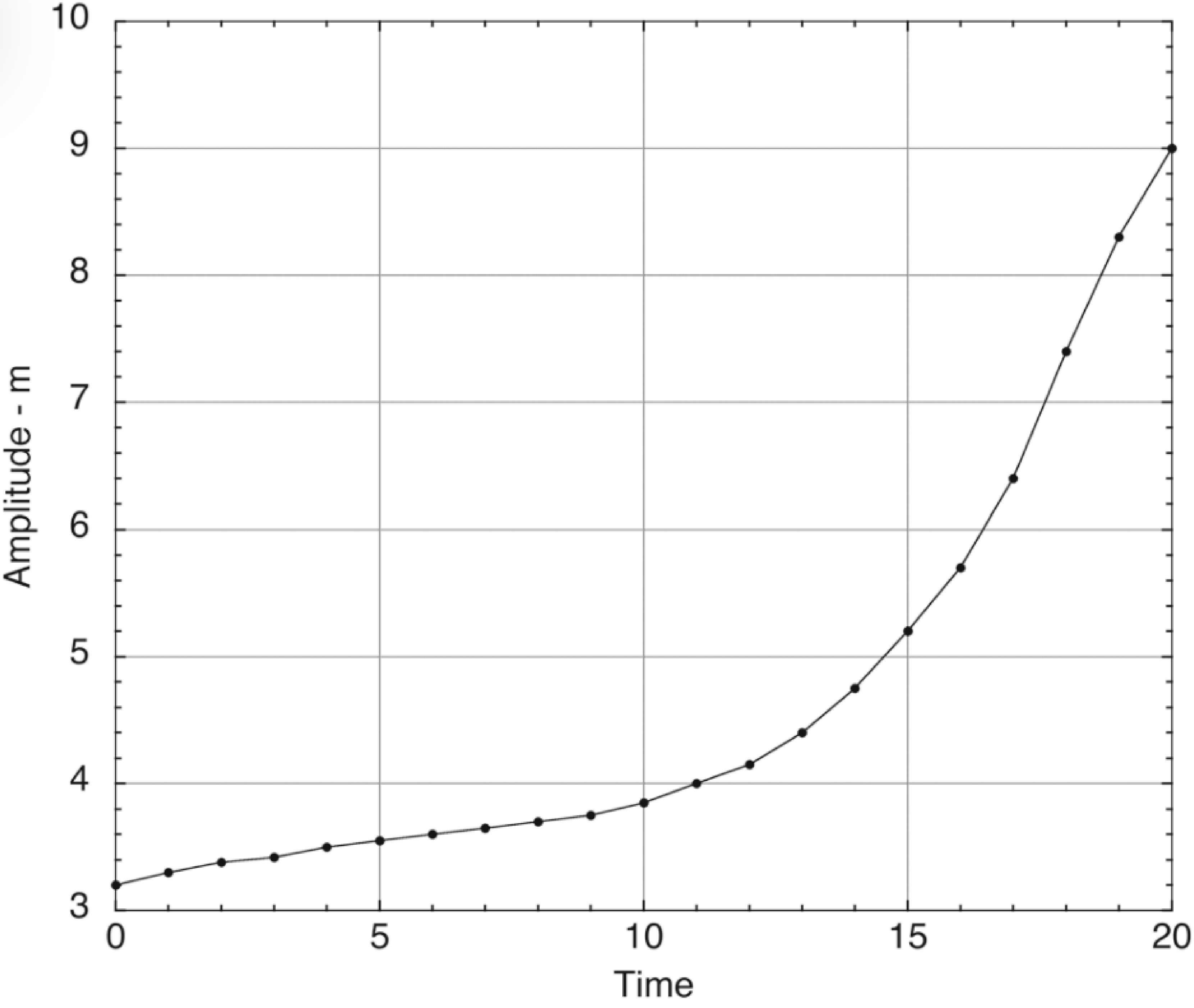
- The ***Traditional View***: The random phase, Gaussian approximation describes ocean surface waves
- The ***Problem is***: This is untrue!

The Wavenumber Plane: Nonlinear Schroedinger, Zakharov Equation and Phillips Figure 8



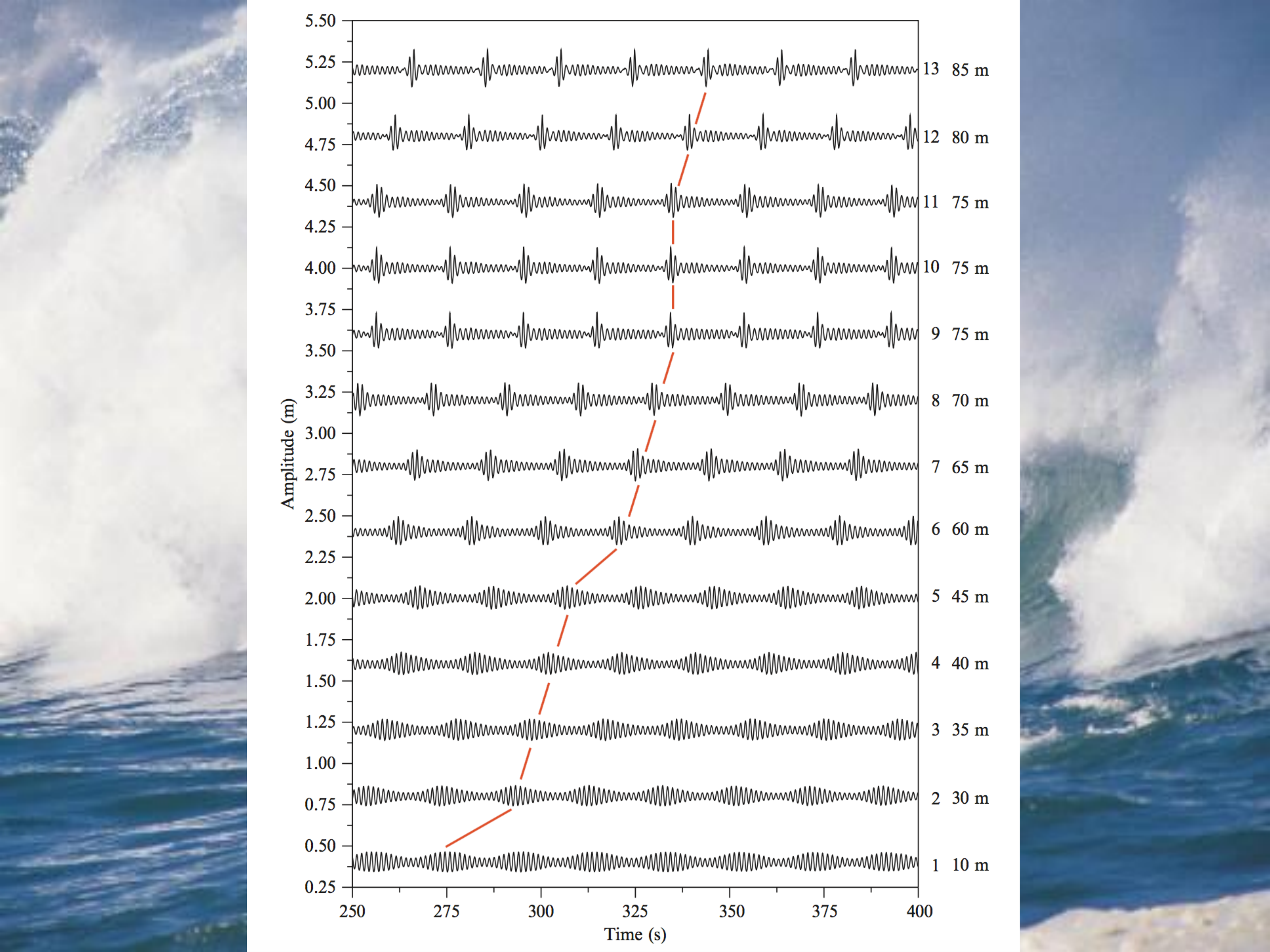

What is a Breather?

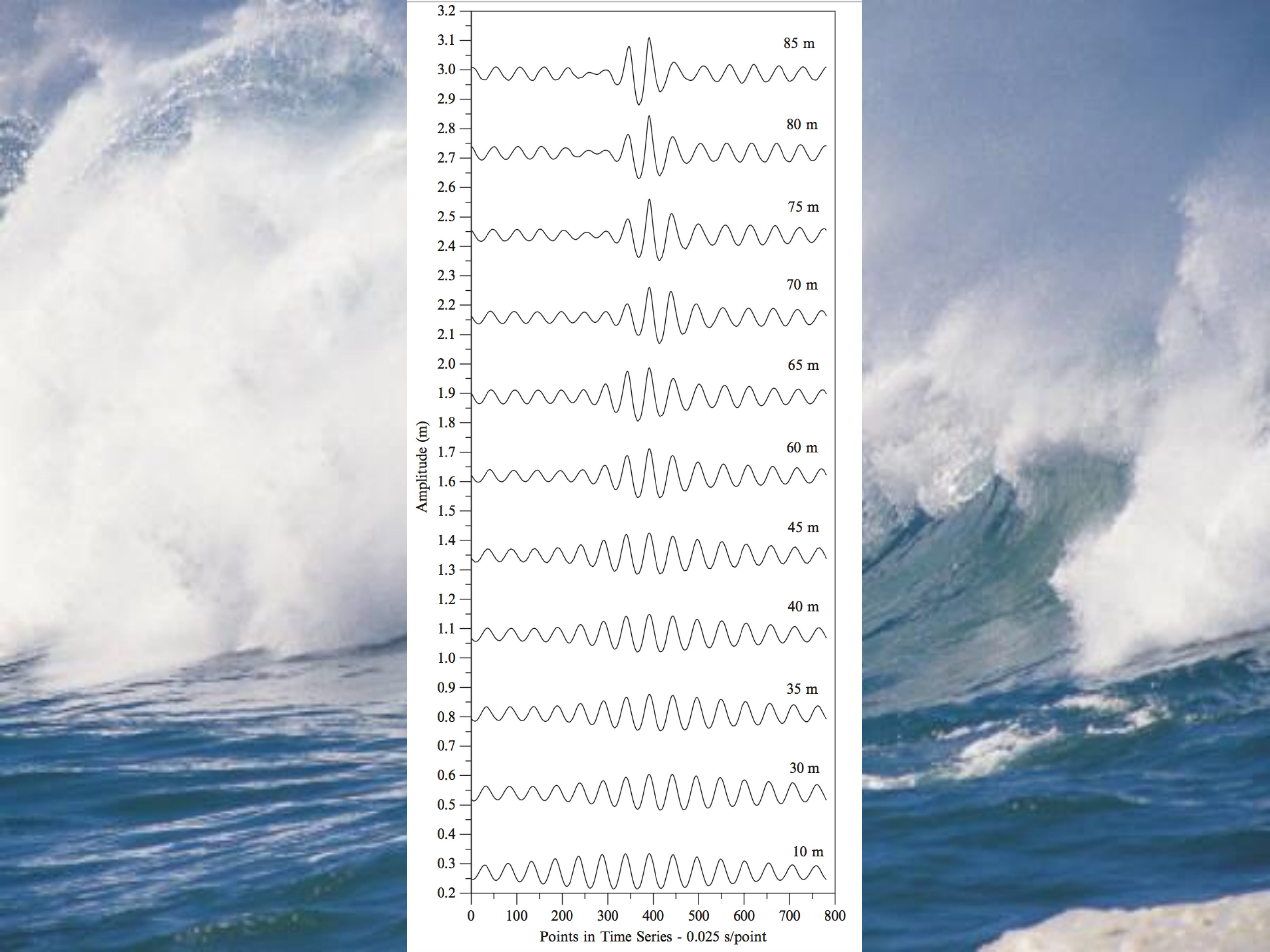

Amplitude – m
t – sec
Amplitude – m
t – sec
Amplitude – m
t – sec



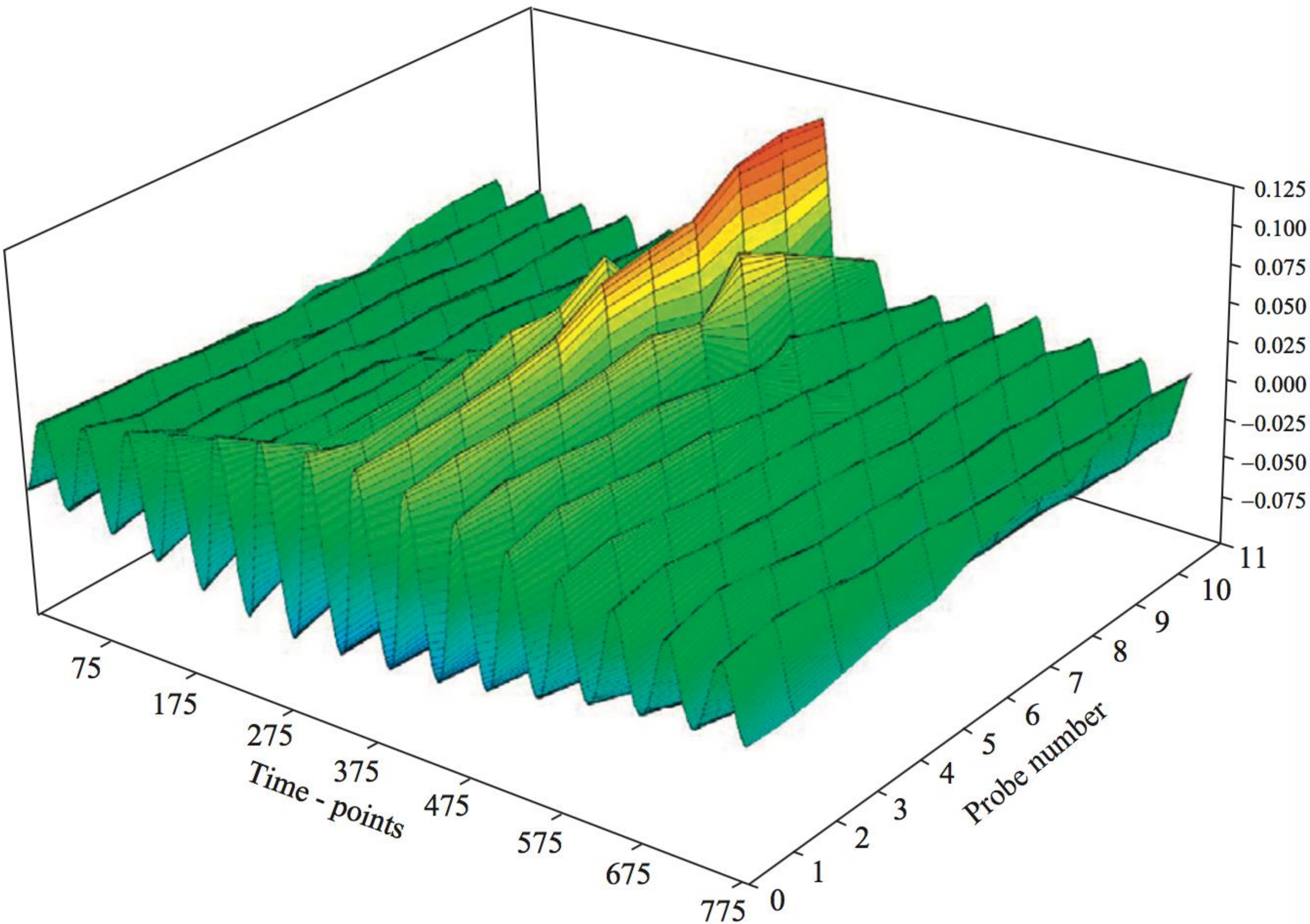
Amplitude - m
Time
10
9
8
7
6
5
4
3
0
5
10
15
20

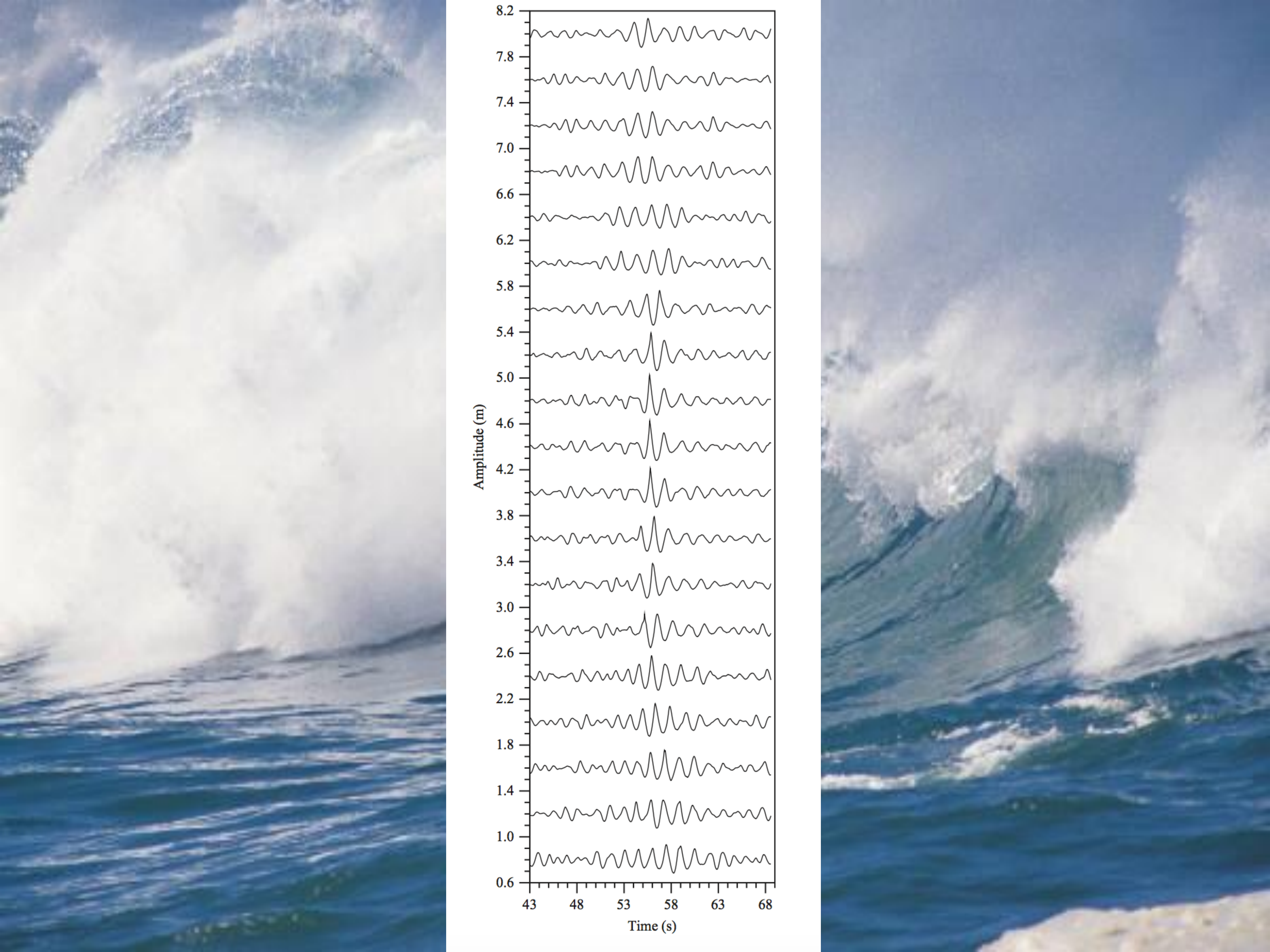

Sintef (Marintek)
2003

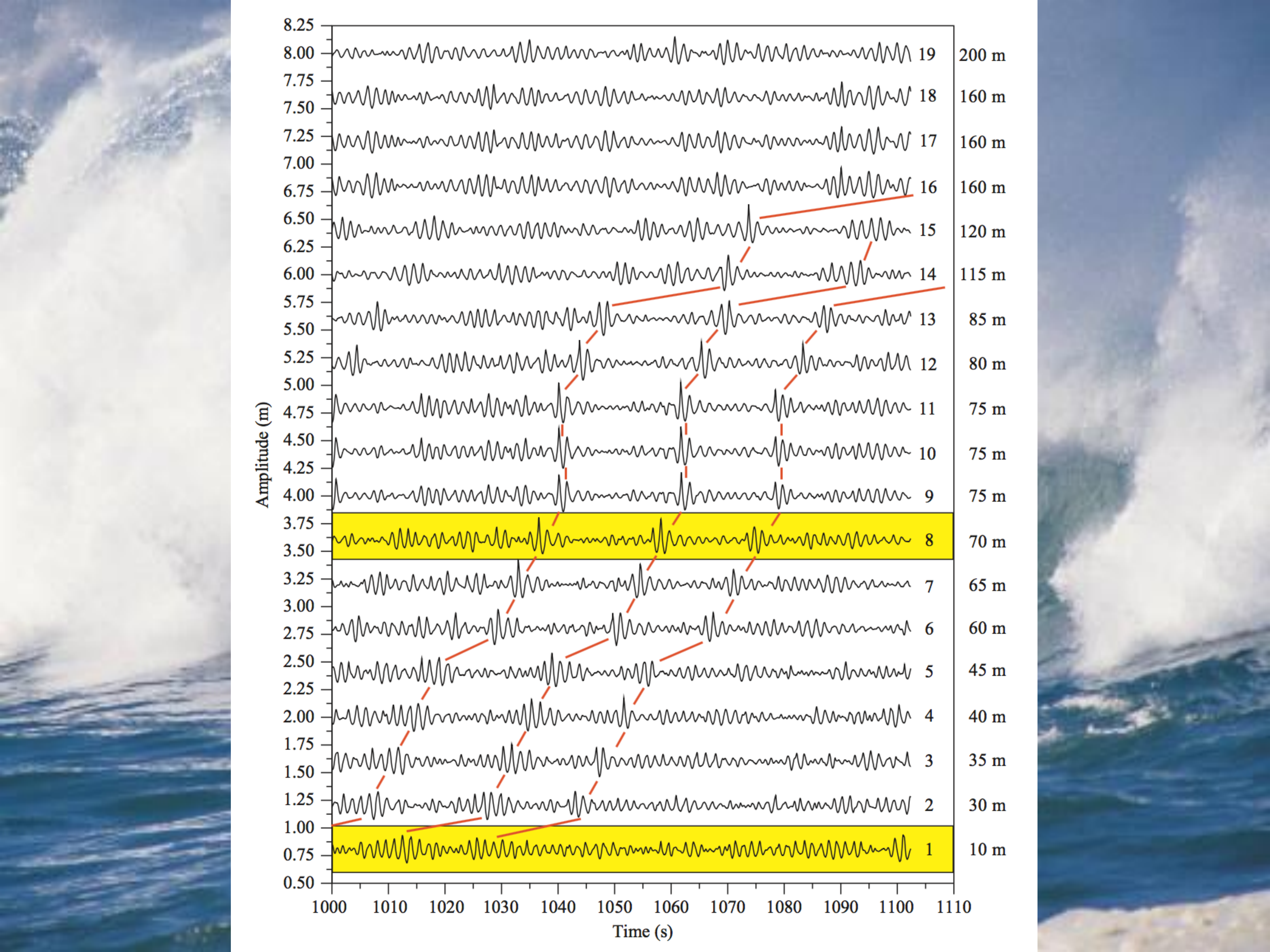

Amplitude (m)
5.50
5.25
5.00
4.75
4.50
4.25
4.00
3.75
3.50
3.25
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
13 85 m
12 80 m
11 75 m
10 75 m
9 75 m
8 70 m
7 65 m
6 60 m
5 45 m
4 40 m
3 35 m
2 30 m
1 10 m
250
275
300
325
350
375
400
Time (s)

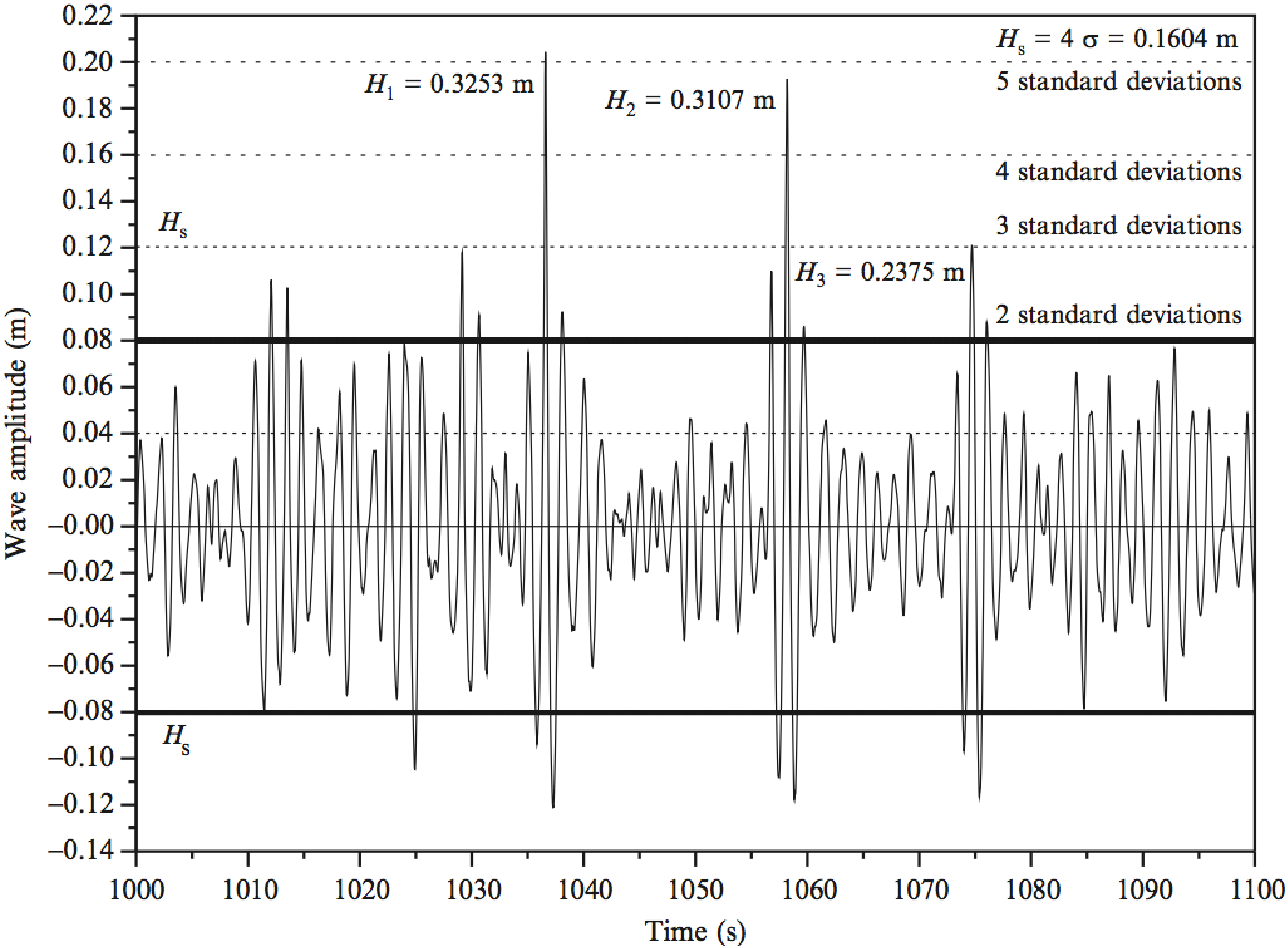


85 m
80 m
75 m
70 m
65 m
60 m
45 m
40 m
35 m
30 m
10 m
Amplitude (m)
3.2
3.1
3.0
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
100
200
300
400
500
600
700
800
Points in Time Series - 0.025 s/point

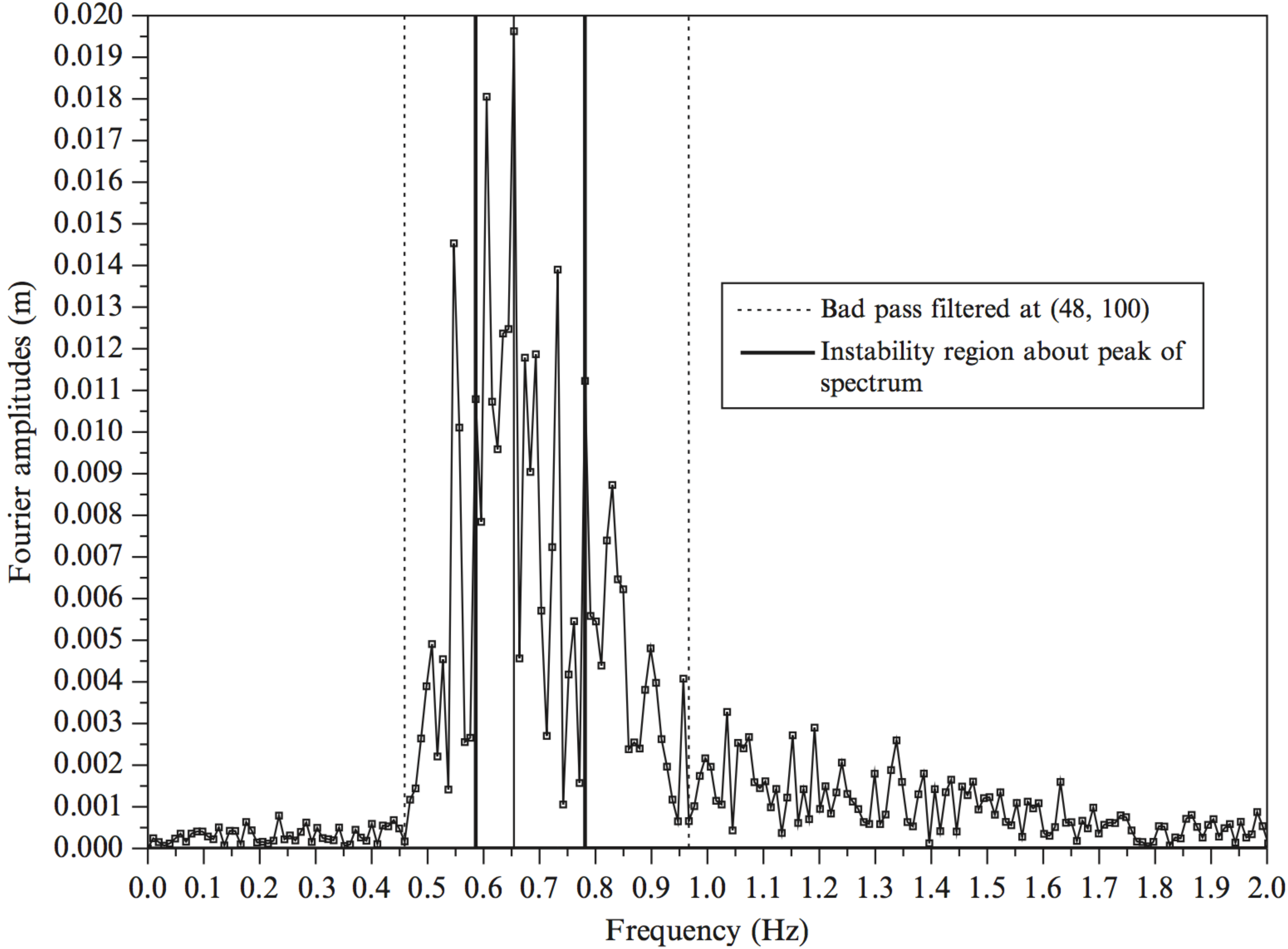

0.125
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
75
175
275
375
475
575
675
775
Time - points
0
1
2
3
4
5
6
7
8
9
10
11
Probe number


8.2
7.8
7.4
7.0
6.6
6.2
5.8
5.4
5.0
4.6
4.2
3.8
3.4
3.0
2.6
2.2
1.8
1.4
1.0
0.6
Amplitude (m)
43
48
53
58
63
68
Time (s)

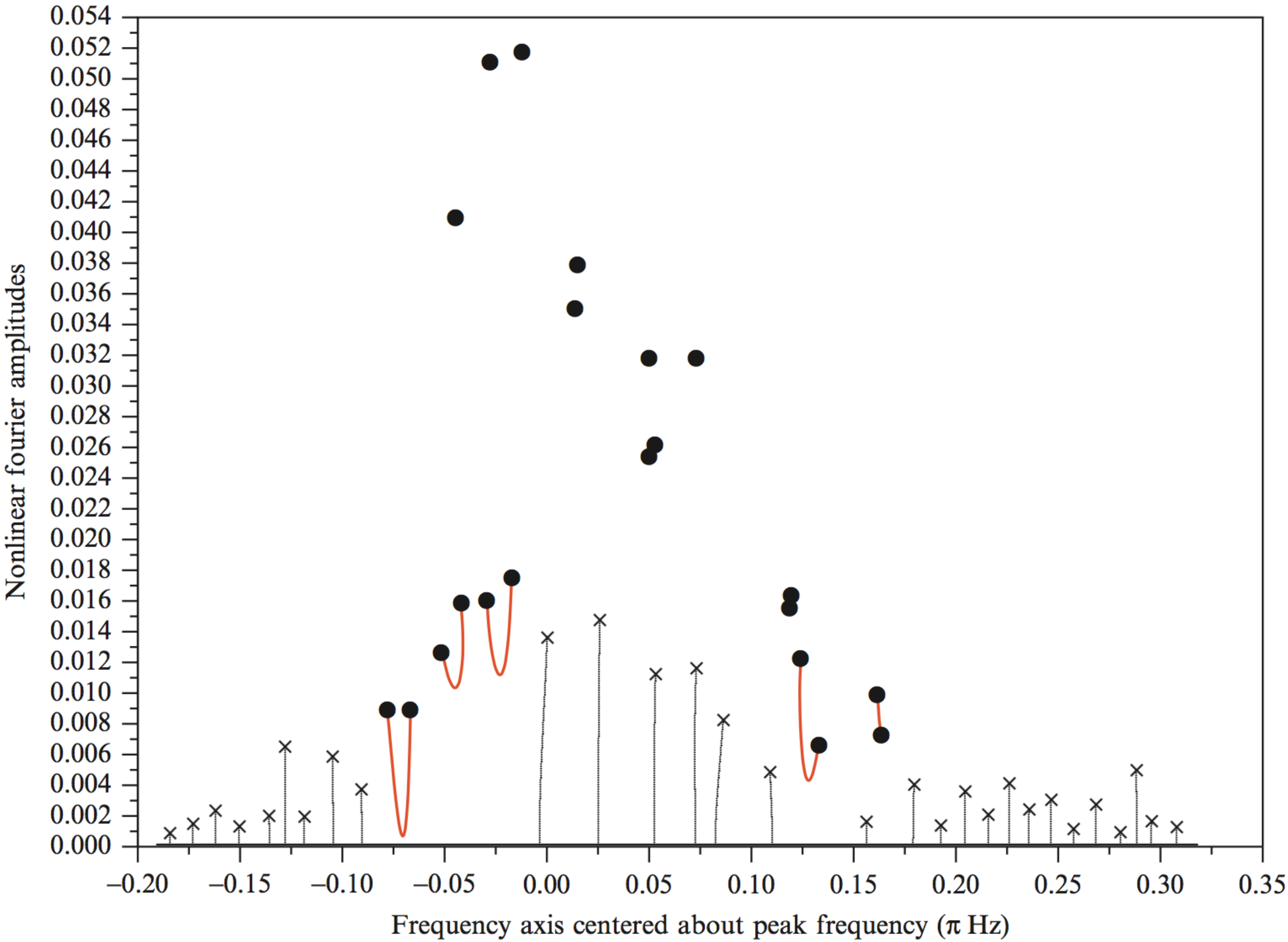

Amplitude (m)
Time (s)
8.25
8.00
7.75
7.50
7.25
7.00
6.75
6.50
6.25
6.00
5.75
5.50
5.25
5.00
4.75
4.50
4.25
4.00
3.75
3.50
3.25
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
1000
1010
1020
1030
1040
1050
1060
1070
1080
1090
1100
1110
19 200 m
18 160 m
17 160 m
16 160 m
15 120 m
14 115 m
13 85 m
12 80 m
11 75 m
10 75 m
9 75 m
8 70 m
7 65 m
6 60 m
5 45 m
4 40 m
3 35 m
2 30 m
1 10 m



$H_s = 4\ \sigma = 0.1604$ m
5 standard deviations
4 standard deviations
3 standard deviations
2 standard deviations
$H_1 = 0.3253$ m
$H_2 = 0.3107$ m
$H_3 = 0.2375$ m
H_s
H_s
Wave amplitude (m)
Time (s)
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
−0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
1000
1010
1020
1030
1040
1050
1060
1070
1080
1090
1100

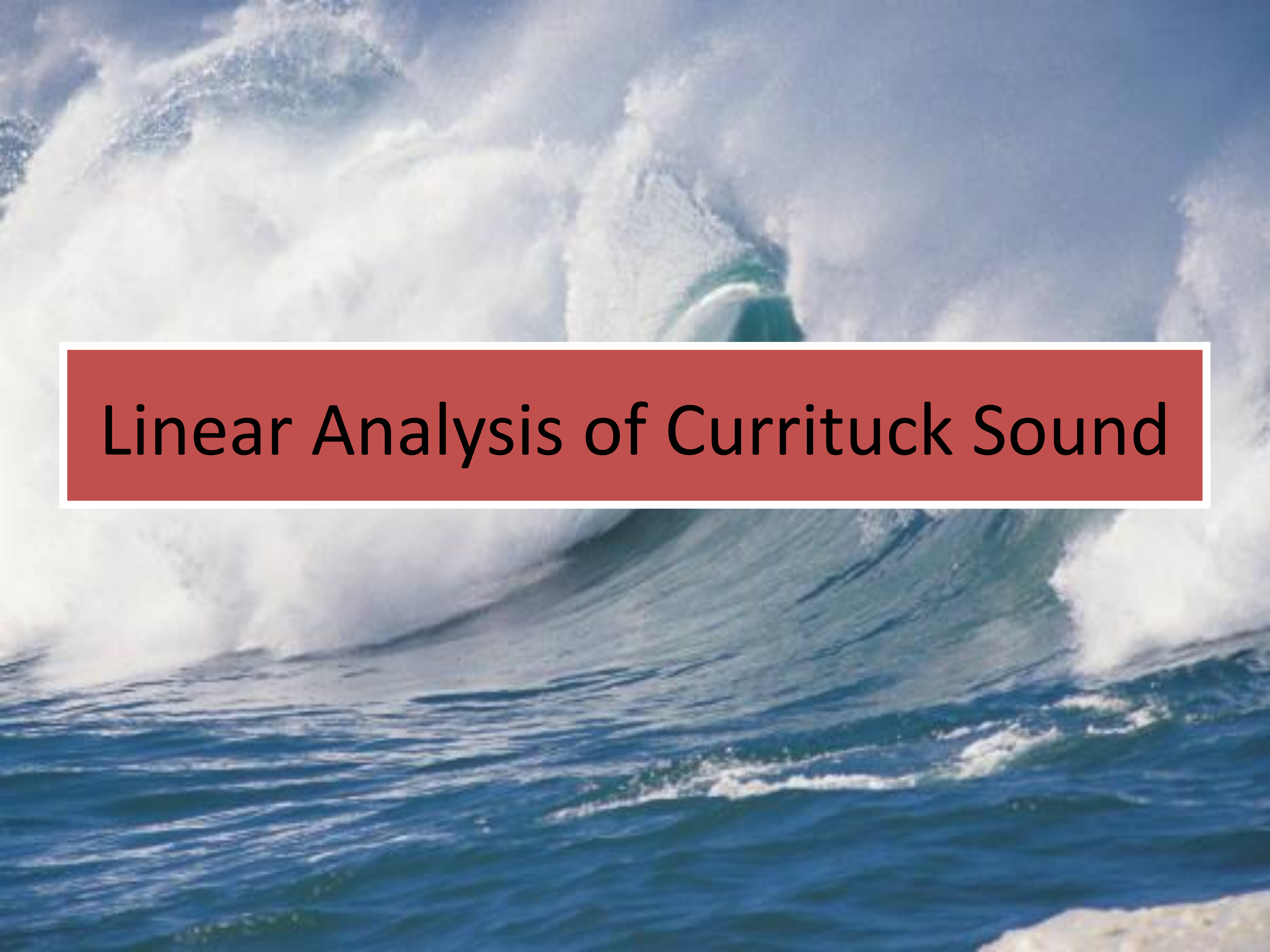


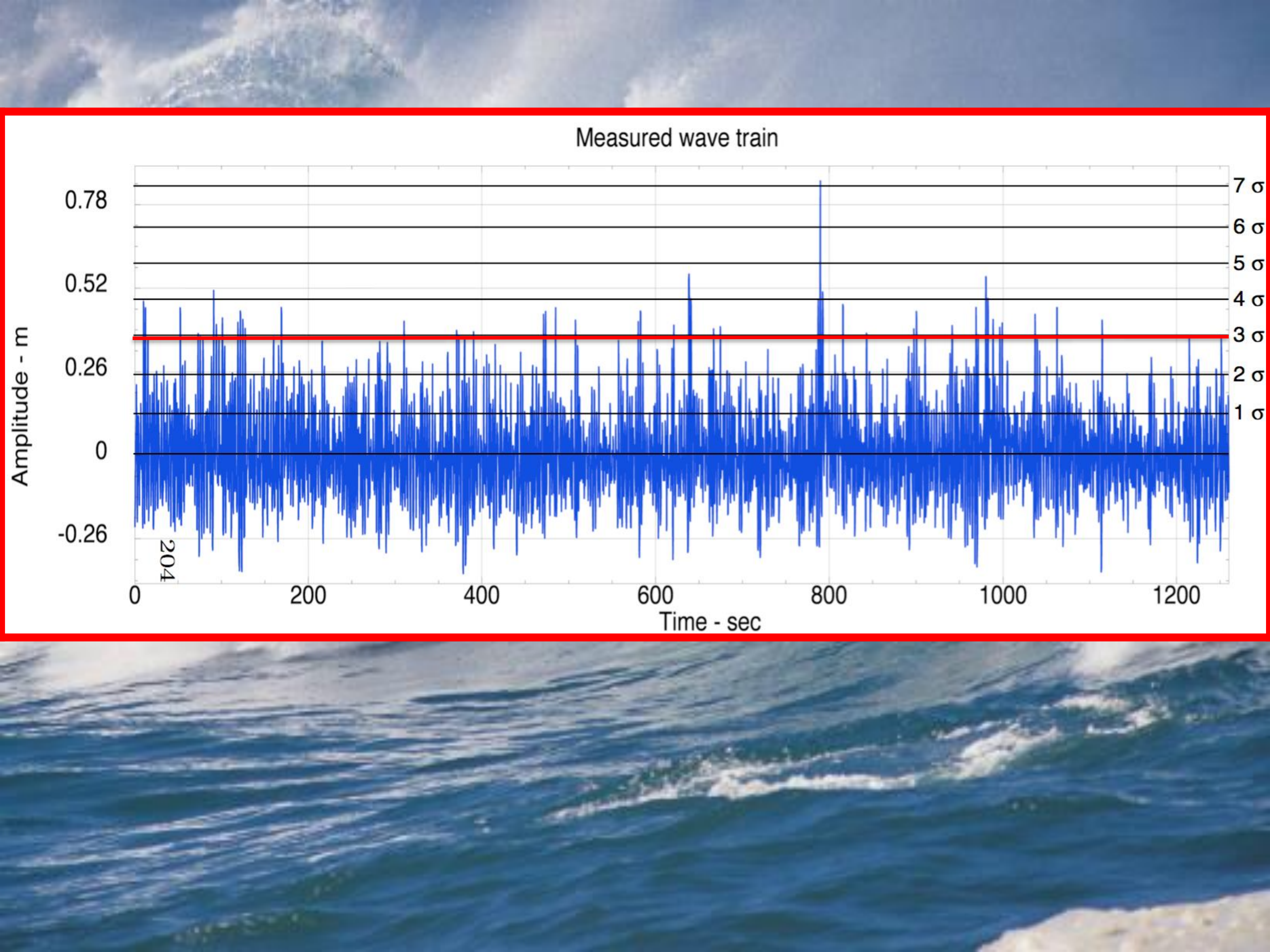
Fourier amplitudes (m)
0.020
0.019
0.018
0.017
0.016
0.015
0.014
0.013
0.012
0.011
0.010
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.000
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0
Frequency (Hz)
Bad pass filtered at (48, 100)
Instability region about peak of spectrum



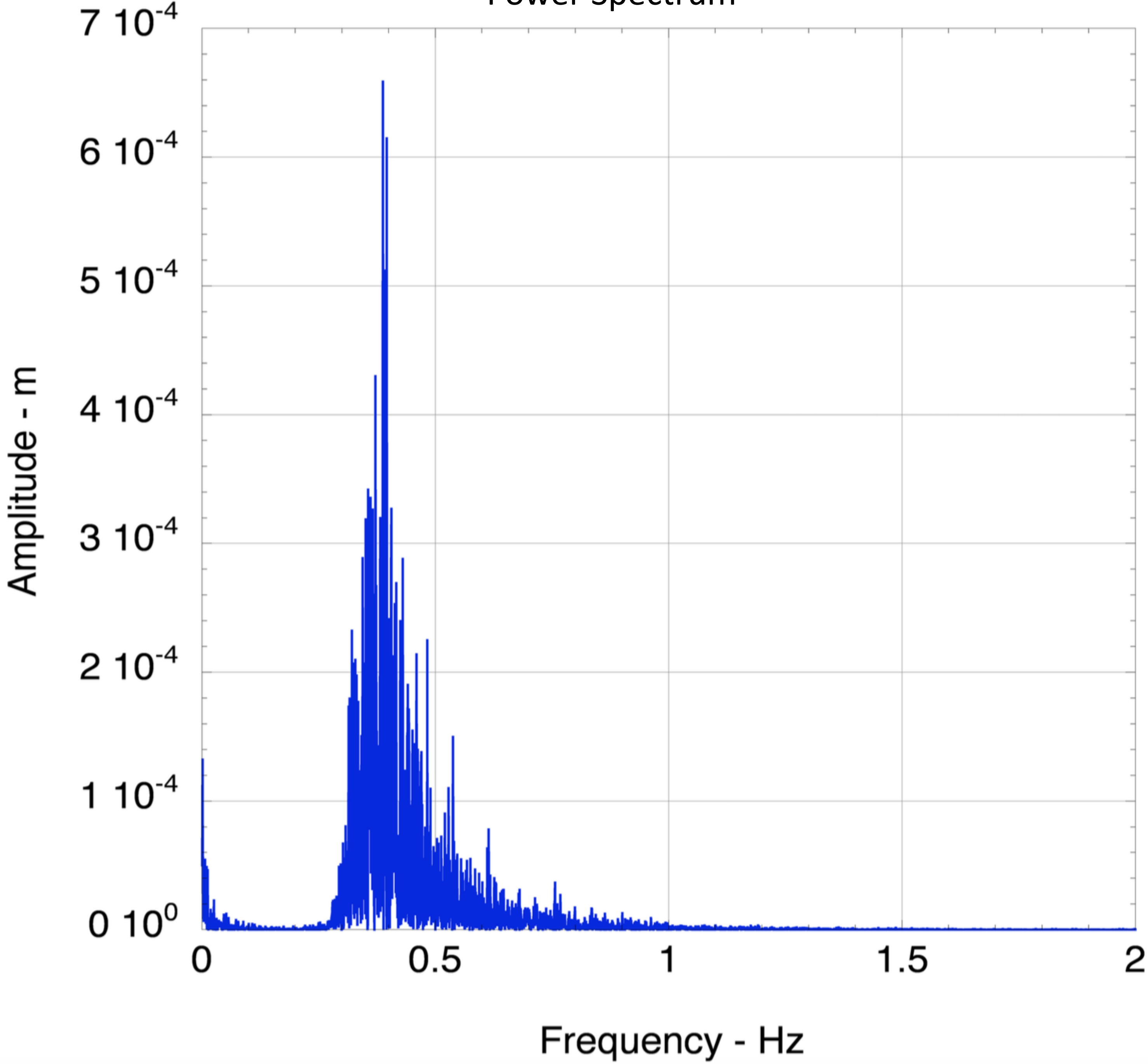
Nonlinear fourier amplitudes
0.054
0.052
0.050
0.048
0.046
0.044
0.042
0.040
0.038
0.036
0.034
0.032
0.030
0.028
0.026
0.024
0.022
0.020
0.018
0.016
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
−0.20
−0.15
−0.10
−0.05
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
Frequency axis centered about peak frequency (π Hz)

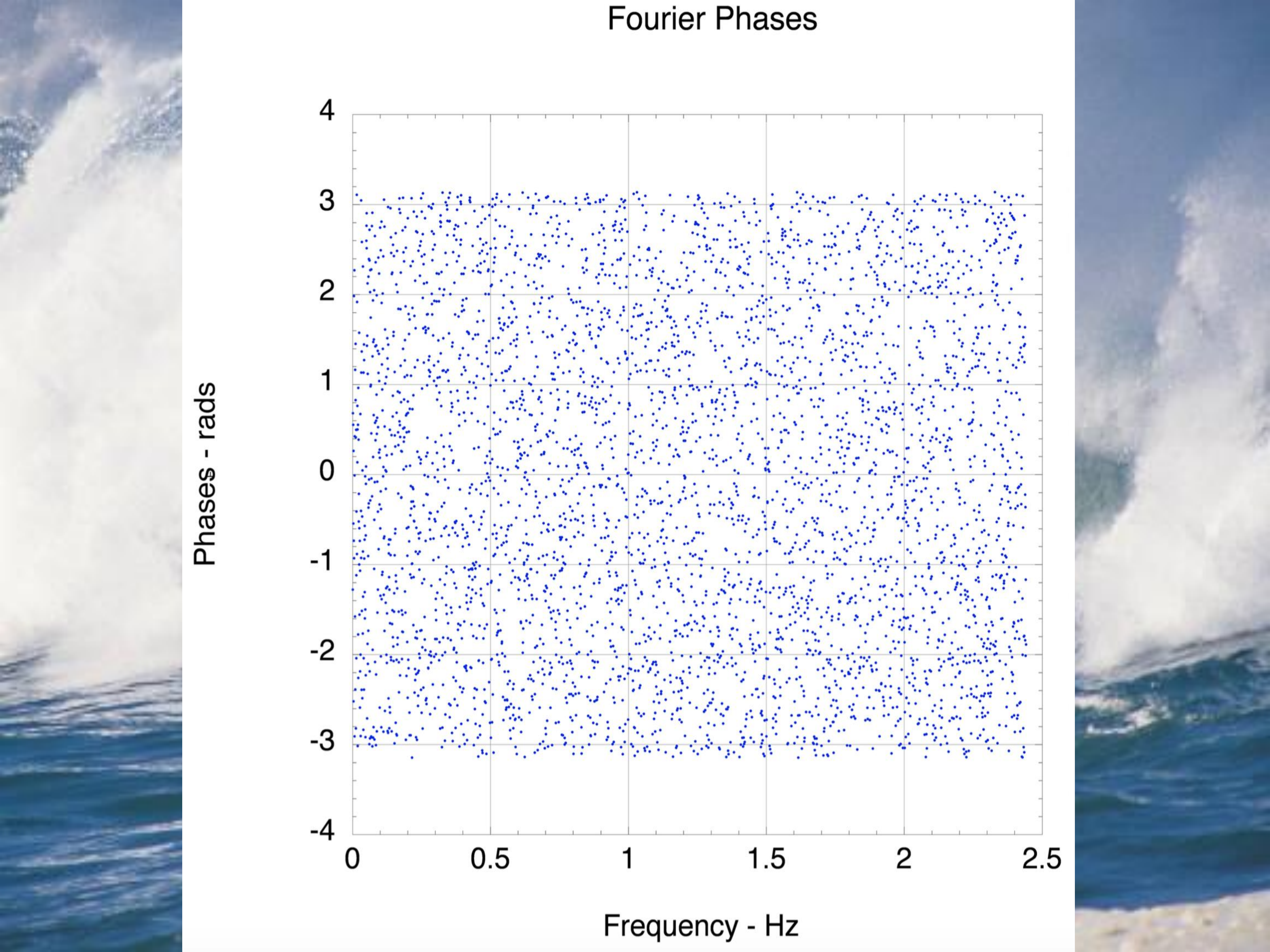
Linear Analysis of Currituck Sound

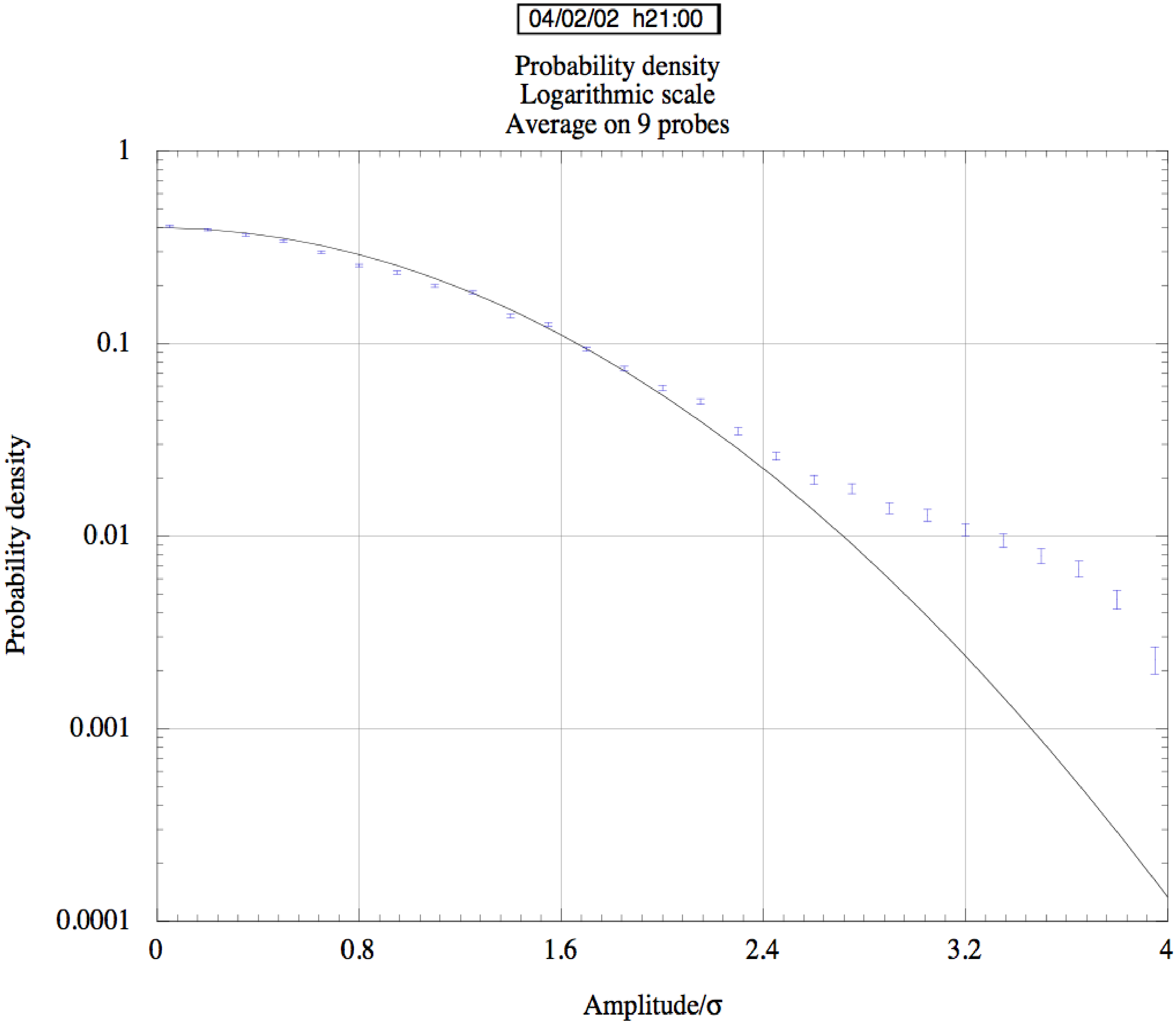



Measured wave train
Amplitude - m
0.78
0.52
0.26
0
-0.26
7 σ
6 σ
5 σ
4 σ
3 σ
2 σ
1 σ
0
200
400
600
800
1000
1200
Time - sec
204

Power Spectrum



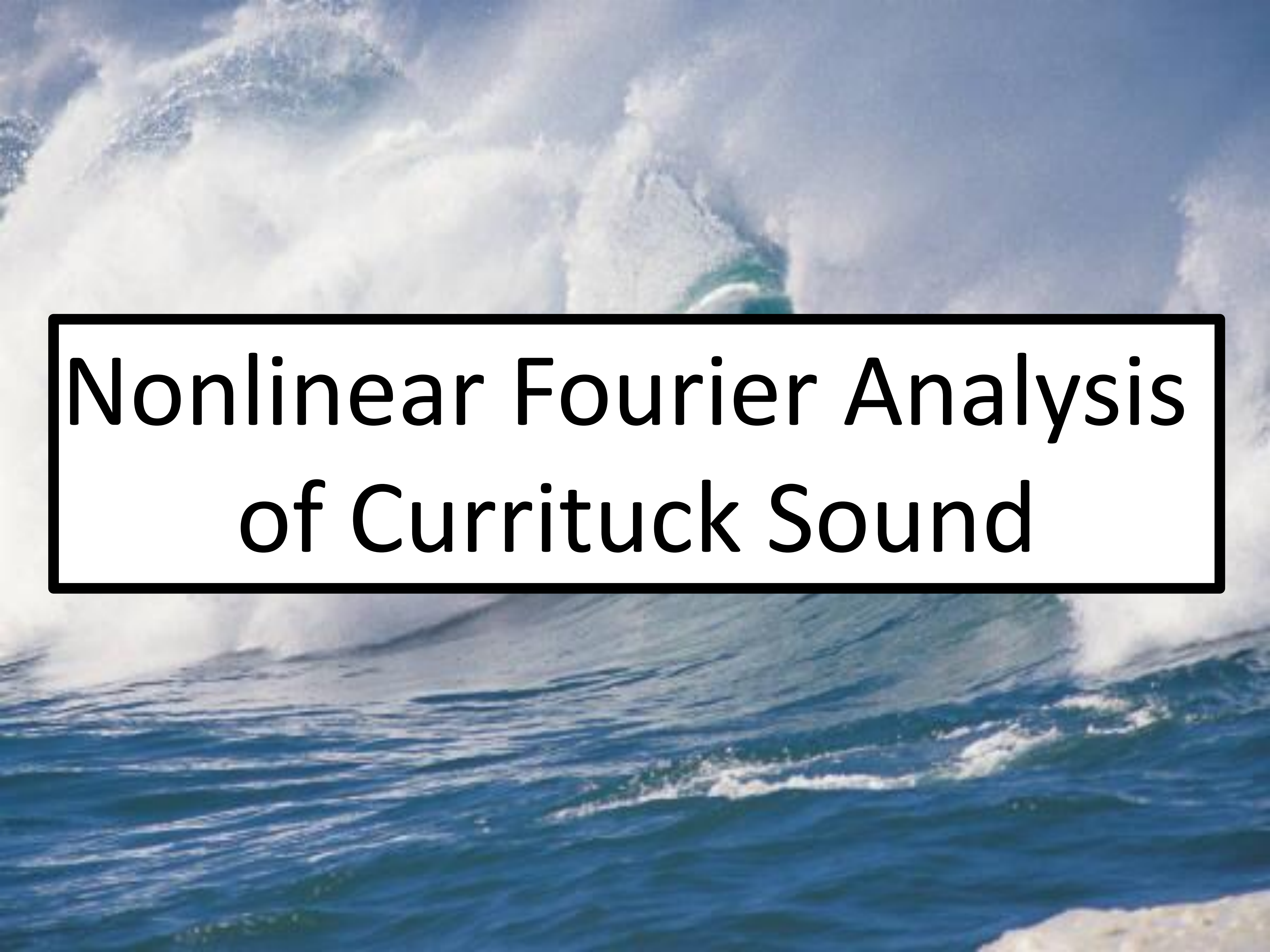

Fourier Phases
Phases - rads
4
3
2
1
0
-1
-2
-3
-4
0
0.5
1
1.5
2
2.5
Frequency - Hz


04/02/02 h21:00
Probability density
Logarithmic scale
Average on 9 probes
Probability density
1
0.1
0.01
0.001
0.0001
0
0.8
1.6
2.4
3.2
4
Amplitude/σ

Conclusions: The Traditional View

- Ocean waves are a ***near linear Random Process*** whose amplitudes are ***Gaussian*** [Kinsman’s book].
- If the spectrum is “narrow banded” then the envelope (modulation) is ***Rayleigh*** [Longuet-Higgins, 1955], implying that the wave heights are also Rayleigh.
- The probability “tail” at high amplitude is due to the ***Stokes correction***, which makes the waves higher and steeper.

Nonlinear Fourier Analysis of Currituck Sound



Soliton Turbulence in Shallow Water Ocean Surface Waves

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[3]*Department of Ocean Engineering, University of North Florida, Jacksonville, Florida 32224-7699, USA*
[4]*Dipartimento di Fisica, Universitá di Torino, Torino 10125, Italy*
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(Received 3 March 2014; revised manuscript received 30 June 2014; published 2 September 2014)

We analyze shallow water wind waves in Currituck Sound, North Carolina and experimentally confirm, for the first time, the presence of soliton turbulence in ocean waves. Soliton turbulence is an exotic form of nonlinear wave motion where low frequency energy may also be viewed as a dense soliton gas, described theoretically by the soliton limit of the Korteweg–deVries equation, a completely integrable soliton system: Hence the phrase "soliton turbulence" is synonymous with "integrable soliton turbulence." For periodic-quasiperiodic boundary conditions the ergodic solutions of Korteweg–deVries are exactly solvable by finite gap theory (FGT), the basis of our data analysis. We find that large amplitude measured wave trains near the energetic peak of a storm have low frequency power spectra that behave as $\sim\omega^{-1}$. We use the linear Fourier transform to estimate this power law from the power spectrum and to filter densely packed soliton wave trains from the data. We apply FGT to determine the soliton spectrum and find that the low frequency $\sim\omega^{-1}$ region is soliton dominated. The solitons have random FGT phases, a soliton random phase approximation, which supports our interpretation of the data as soliton turbulence. From the probability density of the solitons we are able to demonstrate that the solitons are dense in time and highly non-Gaussian.

DOI: 10.1103/PhysRevLett.113.108501 PACS numbers: 92.10.Hm, 92.10.Lq, 92.10.Sx

Available online at www.sciencedirect.com

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Procedia IUTAM 26 (2018) 112–123

Procedia IUTAM

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IUTAM Symposium Wind Waves, 4–8 September 2017, London, UK

Nonlinear Fourier Methods for Ocean Waves

Alfred R. Osborne*

Nonlinear Waves Research Corporation, Alexandria, VA 22314, U. S. A.

Abstract

Multiperiodic Fourier series solutions of *integrable nonlinear wave equations* are applied to the study of ocean waves for scientific and engineering purposes. These series can be used to compute analytical formulae for the *stochastic properties* of nonlinear equations, in analogy to the standard approach for linear equations. Here I emphasize analytically computable results for the *correlation functions*, *power spectra* and *coherence functions* of a *nonlinear random process* associated with an integrable nonlinear wave equation. The multiperiodic Fourier series have the advantage that the *coherent structures* of soliton physics are encoded in the formulation, so that *solitons*, *breathers*, *vortices*, etc. are contained in the *temporal evolution* of the nonlinear power spectrum and phases. I illustrate the method for the Korteweg-deVries and nonlinear Schrödinger equations. Applications of the method to the analysis of data are discussed.

Peer-review under responsibility of the scientific committee of the IUTAM Symposium Wind Waves.

Keywords: nonlinear integrable wave equations; finite gap theory; periodic inverse scattering transform; nonlinear Fourier analysis; nonlinear ocean waves; nonlinear numerical methods; nonlinear time series analysis.

Ocean Dynamics
https://doi.org/10.1007/s10236-018-1232-y

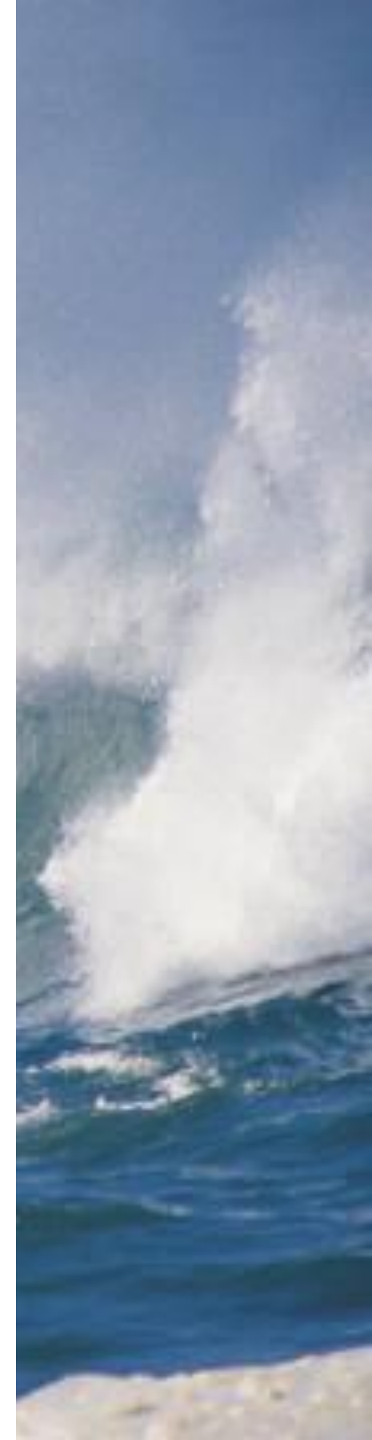
Highly nonlinear wind waves in Currituck Sound: dense breather turbulence in random ocean waves

Alfred R. Osborne[1] · Donald T. Resio[2] · Andrea Costa[3,4] · Sonia Ponce de León[5] · Elisabetta Chirivì[6]

Received: 21 May 2018 / Accepted: 23 October 2018

Abstract

We analyze surface wave data taken in Currituck Sound, North Carolina, during a storm on 4 February 2002. Our focus is on the application of *nonlinear Fourier analysis* (NLFA) methods (Osborne 2010) to analyze the data set: The approach spectrally decomposes a nonlinear wave field into *sine waves*, *Stokes waves*, and *phase-locked Stokes waves* otherwise known as *breather trains*. Breathers are nonlinear beats, or packets which "breathe" up and down smoothly over *cycle times* of minutes to hours. The maximum amplitudes of the packets during the cycle have a largest central wave whose properties are often associated with the study of "rogue waves." The mathematical physics of the nonlinear Schrödinger (NLS) equation is assumed and the methods of algebraic geometry are applied to give the *nonlinear spectral representation*. The distinguishing characteristic of the NLFA method is its ability to spectrally decompose a time series into its *nonlinear coherent structures* (Stokes waves and breathers) rather than just sine waves. This is done by the implementation of *multidimensional, quasi-periodic Fourier series*, rather than ordinary Fourier series. To determine preliminary estimates of nonlinearity, we use the significant wave height H_s, the peak period T_p, and the length of the time series T. The time series analyzed here have 8192 points and T =1677.72 s = 27.96 min. Near the peak of the storm, we find $H_s \approx 0.55$ m, $T_p \approx 2.4$ s so that for the wave steepness of a near Gaussian process, $S = \left(\pi^{5/2}/g\right) H_s/T_p^2$, we find $S \approx 0.17$, quite high for ocean waves. Likewise, we estimate the Benjamin-Feir (BF) parameter for a near Gaussian process, $I_{BF} = \left(\pi^{5/2}/g\right) H_s T/T_p^3$, and we find $I_{BF} \approx 119$. Since the BF parameter describes the nonlinear behavior of the *modulational instability*, leading to the formation of breather packets in a measured wave train, we find the I_{BF} for these storm waves to be a surprisingly high number. This is because I_{BF}, as derived here, roughly estimates the number of breather trains in a near Gaussian time series. The BF parameter suggests that there are roughly 119 breather trains in a time series of length 28 min near the peak of the storm, meaning that we would have average breather packets of about 14 s each with about 5-6 waves in each packet. Can these surprising results, estimated from simple parameters, be true from the point of view of the complex nonlinear wave dynamics of the BF instability and the NLS equation? We analyze the data set with the NLFA to verify, from a *nonlinear spectral point of view*, the presence of large numbers of breather trains and we determine many of their properties, including the *rise time* for the breathers to grow to their maximum amplitudes from a quiescent initial state. Energetically, about 95% of the NLFA components are found to consist of breather trains; the remaining small amplitude components are sine and Stokes waves. The presence of a large number of densely packed breather trains suggests an interpretation of the data in terms of *breather turbulence*, highly nonlinear *integrable turbulence* theoretically predicted for the NLS equation, providing an interesting paradigm for the nonlinear wave motion, in contrast to the random phase Gaussian approximation often considered in the analysis of data.



fluids

MDPI

Article

Breather Turbulence: Exact Spectral and Stochastic Solutions of the Nonlinear Schrödinger Equation

Alfred R. Osborne [1,*]

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Abstract: I address the problem of *breather turbulence in ocean waves* from the point of view of the exact *spectral solutions* of the *nonlinear Schrödinger (NLS) equation* using two tools of mathematical physics: (1) the *inverse scattering transform* (IST) for *periodic/quasiperiodic boundary conditions* (also referred to as finite gap theory (FGT) in the Russian literature) and (2) *quasiperiodic Fourier series*, both of which enhance the physical and mathematical understanding of complicated nonlinear phenomena in water waves. The basic approach I refer to as *nonlinear Fourier analysis* (NLFA). The formulation describes wave motion with *spectral components* consisting of *sine waves*, *Stokes waves* and *breather packets* that nonlinearly interact pair-wise with one another. This contrasts to the simpler picture of standard Fourier analysis in which one linearly superposes sine waves. *Breather trains* are *coherent wave packets* that "breath" up and down during their lifetime "cycle" as they propagate, a phenomenon related to *Fermi-Pasta-Ulam (FPU) recurrence*. The central wave of a breather, when the packet is at its maximum height of the FPU cycle, is often treated as a kind of *rogue wave*. *Breather turbulence* occurs when the number of breathers in a measured time series is large, typically several hundred per hour. Because of the prevalence of rogue waves in breather turbulence, I call this exceptional type of sea state a *breather sea* or *rogue sea*. Here I provide theoretical tools for a physical and dynamical understanding of the recent results of Osborne et al [43] in which *dense breather turbulence* was found in experimental surface wave data in Currituck Sound, North Carolina. Quasiperiodic Fourier series are important in the study of ocean waves because they provide a simpler theoretical interpretation and faster numerical implementation of the NLFA, with respect to the IST, particularly with regard to determination of the breather spectrum and their associated phases that are here treated in the so-called *nonlinear random phase approximation*. The actual material developed here focuses on results necessary for the analysis and interpretation of shipboard/offshore platform radar scans and for airborne lidar and synthetic aperture radar (SAR) measurements.

What is Linear Fourier Analysis?

- A Fourier Series:

$$h(x,t)=\sum_{n=-\infty}^{\infty} h_n e^{ik_n x - i\omega_n t}$$

- A ***linear superposition*** of ***sine waves***.
- We get the FFT and so have ***spectra***, ***power spectra***, ***coherence functions***, ***correlation functions***, and all the useful stuff that come from Fourier methods!
- ***Random phase approximation***.

What is ***Nonlinear*** Fourier Analysis?

- A ***quasiperiodic*** ***(multiperiodic) Fourier Series***:

$$\eta(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} \eta_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}}$$

- Does everything that linear Fourier does, but has ***all infinity harmonics***: A ***discretuum***.
- The ***nonlinear Fourier components*** are ***sine waves***, ***Stokes waves***, ***breathers***, ***superbreathers***, etc.
- ***Nonlinear random phase approximation***.

The 2+1 Nonlinear Schroedinger Equation

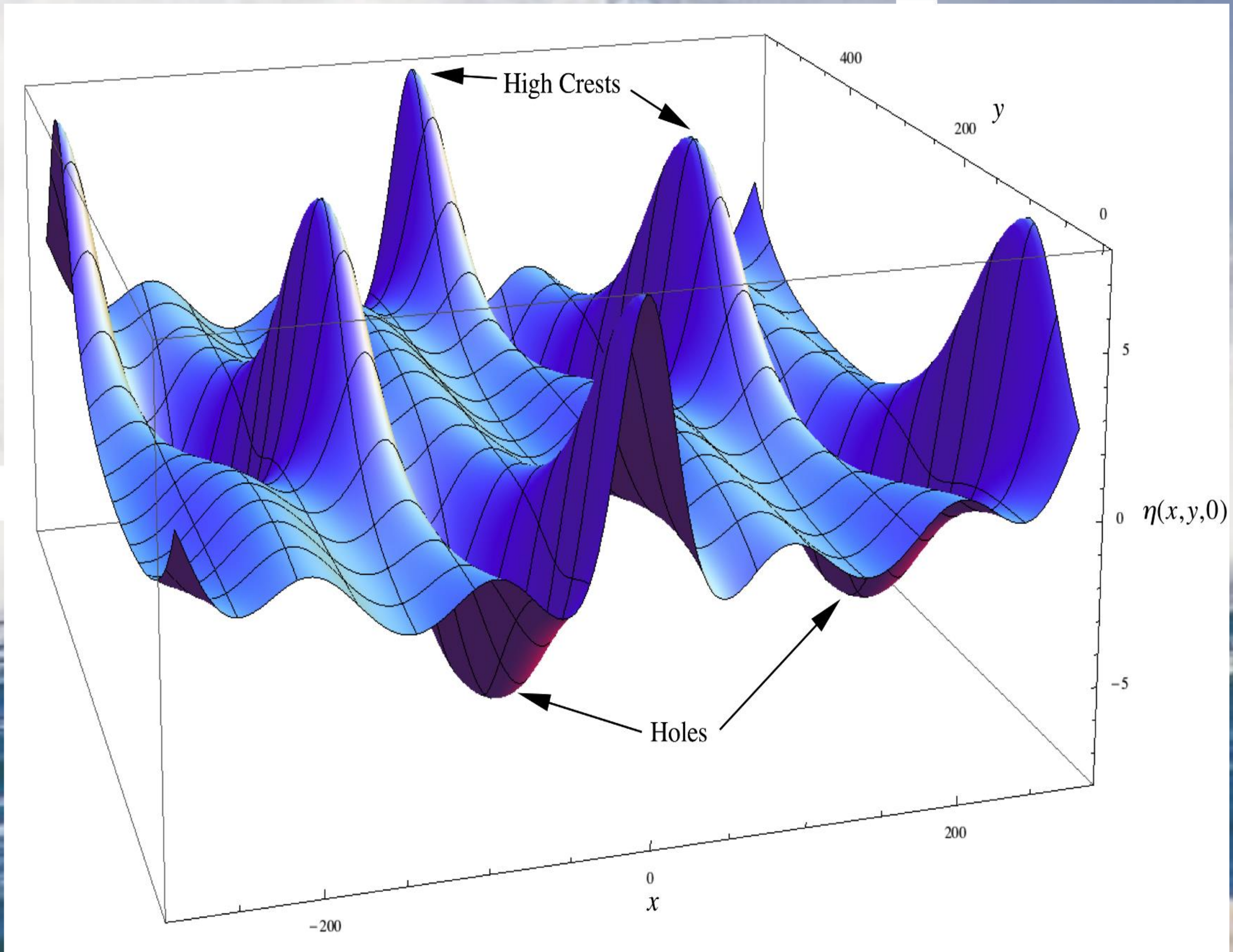
$$iy_t + y_{xx} + s y_{yy} + U(x,y,t)y = 0$$

$$s = \pm 1$$

Depending on the form of the potential the above equation has the form of:

- The ***2+1 NLS equation***
- The ***Dysthe equation***
- The ***extended Dysthe equation***
- The ***Zakharov Equation***

Breather Moving at 30 Degree Angle With Respect To Dominant Wave Direction



Solving the Schroedinger Equation for Arbitrary Potential

$$iy_t + y_{xx} + s y_{yy} + U(x,y,t)y = 0$$

$$s = \pm 1$$

$$y(x,y,t) = \frac{G(x,y,t)}{F(x,y,t)}$$

$$G(x,y,t) \approx a_o \theta(x,y,t \mid \tau, \phi^-) e^{-i\Omega' t}$$

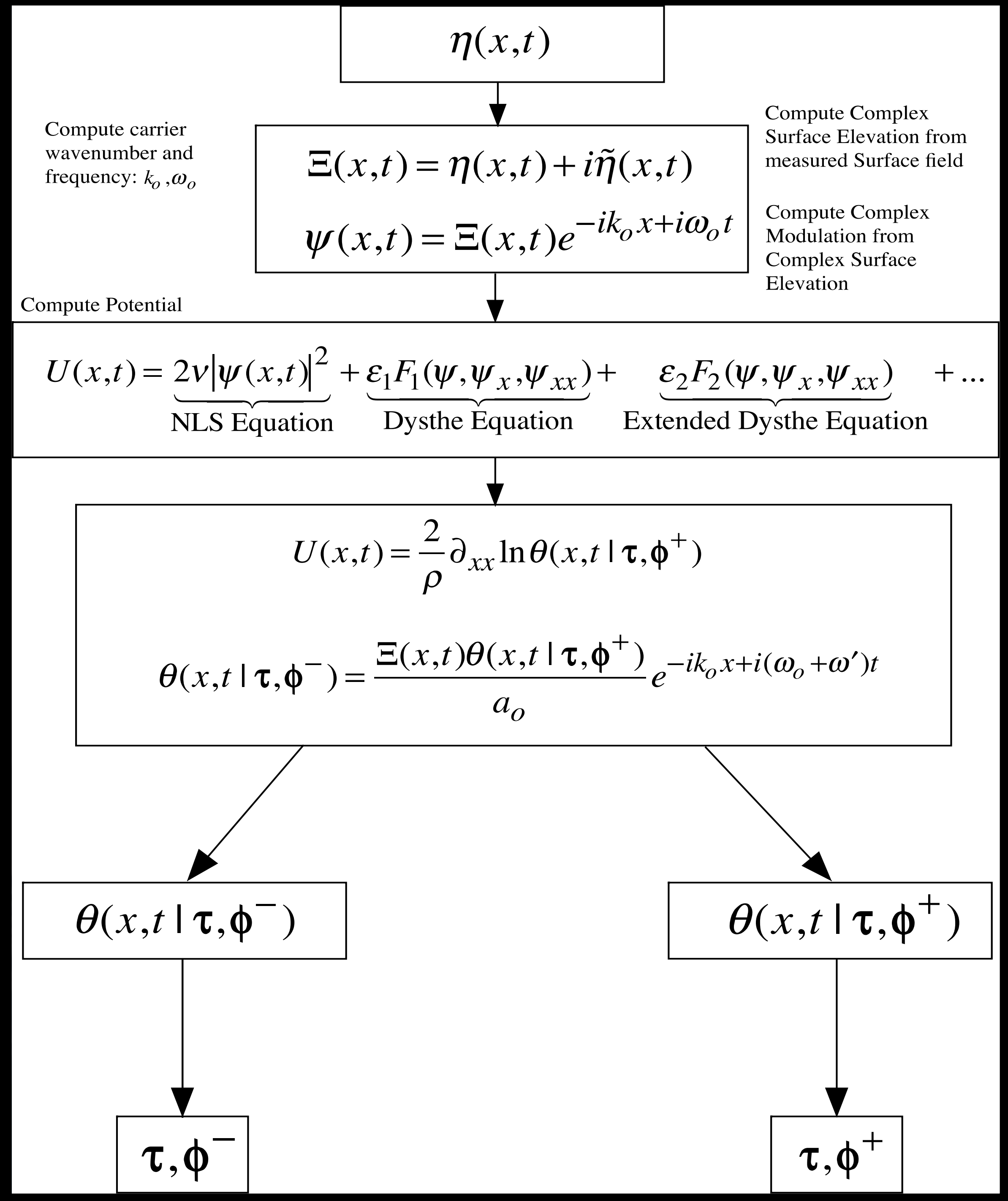
$$F(x,y,t) \approx \theta(x,y,t \mid \tau, \phi^+)$$

$$U(x,y,t) = 2\P_{xx} \ln F(x,y,t) + 2s\P_{yy} \ln F(x,y,t)$$

$$i\left(FG_t - GF_t\right) + \left(FG_{xx} + GF_{xx} - 2F_xG_x\right) + s\left(FG_{yy} + GF_{yy} - 2F_yG_y\right) = l\,FG$$

$$\Xi(x,y,t) \approx a_o \frac{\theta(x,y,t \mid \tau, \phi^-)}{\theta(x,y,t \mid \tau, \phi^+)} e^{i(K_x + k_o)x + iK_y y - i(\Omega + \omega_o + \omega')t}$$

How to Analyze Data



Theorem 3 (Baker [1907], Mumford [1984]) - Construction of Single-valued, Multiply-Periodic meromorphic Functions

The most general, single-valued, multiply-periodic meromorphic functions of N variables with $2N$ sets of periods (obeying the necessary relations, see Baker [1907], p. 224), can be expressed by means of ***theta functions***. They can be determined in ***only three ways***:

(i) $\partial_{xx} \ln \theta(x,t)$

(ii) $$\frac{\theta(x,t \mid \mathbf{B},\phi^{-})}{\theta(x,t \mid \mathbf{B},\phi^{+})}$$

(iii) $$\frac{\prod_{n=1}^{N} \theta(x - x_n, t \mid \mathbf{B},\phi^{-})}{\prod_{n=1}^{N} \theta(x - x_n, t \mid \mathbf{B},\phi^{+})}$$

where the theta functions have the form

$$\theta(x,t) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \theta_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}}, \quad \theta_{\mathbf{n}} = e^{-\frac{1}{2}\mathbf{n}\cdot\tilde{\mathbf{B}}\mathbf{n}} \qquad (2)$$

and $\tilde{\mathbf{B}}$ is the period matrix. The wavenumbers $\mathbf{k}$, frequencies ω and phases ϕ arise from loop integrals over the eigenvalue spectrum.

Suggested proof of the Solution Theorem: One shows that each of the three forms for constructing ***multiply-periodic, meromorphic functions*** given above using theta functions can be written in terms ***of multidimensional, quasiperiodic Fourier series***. This means that

$$\partial_{xx} \ln \theta(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}}$$

$$\frac{\theta(x,t|\mathbf{B},\boldsymbol{\phi}^-)}{\theta(x,t|\mathbf{B},\boldsymbol{\phi}^+)} = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}}$$

$$\frac{\prod_{n=1}^{N} \theta(x-x_n,t|\mathbf{B},\boldsymbol{\phi}^-)}{\prod_{n=1}^{N} \theta(x-x_n,t|\mathbf{B},\boldsymbol{\phi}^+)} = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}}$$

Where the wavenumbers $\mathbf{k}$, frequencies $\boldsymbol{\omega}$, phases $\boldsymbol{\phi}$ and coefficients $u_{\mathbf{n}}$ are determined from the theta functions in the algebraic-geometric solution of a particular nonlinear wave equation. See example below for the KdV equation.

Comment: The above results are possible because theta functions have remarkable properties and these allow one to discuss ***an algebra of theta functions*** and their derivatives, integrals, i.e. they can be ***added***, ***subtracted***, ***multiplied*** and ***divided***.

Add on the Perturbations

$$\psi(x,t) = a_o \frac{\prod_{n=0}^{N} \theta_n(x,t \mid \tau,\phi^-)}{\prod_{n=0}^{N} \theta_n(x,t \mid \tau,\phi^+)} e^{-\omega' t}$$

$$= a_o \frac{\theta_o(x,t \mid \tau,\phi^-)\theta_1(x,t \mid \tau,\phi^-)\theta_2(x,t \mid \tau,\phi^-)\ldots}{\underbrace{\theta_o(x,t \mid \tau,\phi^+)}_{\substack{\text{NLS} \\ \text{Equation}}}\underbrace{\theta_1(x,t \mid \tau,\phi^+)}_{\substack{\text{Dysthe} \\ \text{Equation}}}\ \underbrace{\theta_2(x,t \mid \tau,\phi^+)}_{\substack{\text{Extended Dysthe} \\ \text{Equation}}}\ \ldots} e^{-\omega' t}$$

Step 1: the Integrable Case 1+1 NLS

Integrable 1+1 NLS as Ratio of Theta Functions, Quasiperiodic Fourier Series and Almost Periodic Fourier Series

Finite Gap Spectral Solutions For Surface Elevation of 1+1 NLS

$$\Xi(x,y,t) = a_o \frac{\theta(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^-)}{\theta(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^+)} e^{i[K+K'+k_o]x - i[\Omega+\Omega'+\omega_o]t}$$

[Kotljarov & Its, 1976], [Tracy & Chen, 1988]

↓

Theta Functions With Two Sets of Phases

$$\theta(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^\mp) = \sum_{\mathbf{n}\in\mathbb{Z}} \theta_{\mathbf{n}} e^{i\mathbf{n}\cdot\boldsymbol{\kappa}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}^\mp}, \quad \theta_{\mathbf{n}} = e^{-i\pi\mathbf{n}\cdot\boldsymbol{\tau}\mathbf{n}}$$

↓

Solution of 1+1 NLS as ***Quasiperiodic Fourier Series***: Most General, Single Valued, Multiply Periodic, Meromorphic Function

$$\Xi(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^\mp) = \sum_{\mathbf{n}\in\mathbb{Z}} \Xi_{\mathbf{n}}(\boldsymbol{\tau}) e^{i\mathbf{n}\cdot\boldsymbol{\kappa}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\Phi}(\boldsymbol{\phi}^\mp)}$$

[Baker, 1897, 2007], [Zygmund, 1935] [Mumford, 1982], [Osborne, 2018, 2019]

↓

Solution For 1+1 NLS as ***Almost Periodic Fourier Series*** with limits $N \to \infty, \quad M \to \infty$.

$$\Xi(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^\mp) = \sum_{n=-\infty}^{\infty} \Xi_n e^{iK_n x - i\Omega_n t + i\Phi_n}$$

[Osborne, 2010]

Step 2:
The Nonintegrable Case 1+1 NLS+ Perturbations

Nonintegrable 1+1 NLS Plus Perturbations as Ratio of Product of Theta Functions, Quasiperiodic Fourier Series and Almost Periodic Fourier Series

Finite Gap Spectral Solutions For 1+1 NLS + Perturbations

$$\Xi(x,t) = a_o \frac{\prod_{m=1}^{M} \theta_m(x,t \mid \boldsymbol{\tau}_n, \boldsymbol{\phi}_n^-)}{\prod_{m=1}^{M} \theta_m(x,t \mid \boldsymbol{\tau}_n, \boldsymbol{\phi}_n^+)} e^{i[K(t)+k_o]x - i[\Omega(t)+\omega_o+\omega']t}$$

[Kotljarov & Its, 1976], [Tracy & Chen, 1988]

↓

Theta Functions With Time Varying Parameters

$$\theta_m(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} \theta_{m,\mathbf{n}}(\boldsymbol{\tau}_n(t)) e^{i\mathbf{n}\bullet\mathbf{k}_m(t)x - i\mathbf{n}\bullet\boldsymbol{\omega}_m(t)t - i\mathbf{n}\bullet\boldsymbol{\phi}_m^{\mp}(t)}, \quad \theta_{m,\mathbf{n}} = e^{\pi i \mathbf{n}\bullet\boldsymbol{\tau}_m(t)\mathbf{n}}$$

↓

Solution of 1+1 NLS + Perturbations as ***Quasiperiodic Fourier Series with time varying parameters***: Single Valued, Multiply Periodic, Meromorphic Functions

$$\Xi(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} \Xi_{\mathbf{n}}(\boldsymbol{\tau}(t)) e^{i\mathbf{n}\bullet\mathbf{k}(t)x - i\mathbf{n}\bullet\boldsymbol{\omega}(t)t - i\mathbf{n}\bullet\boldsymbol{\Phi}(t)}$$

[Baker, 1976], [Mumford, 1982], [Osborne, 2018, 2019]

↓

Solution For 1+1 NLS + Perturbations as ***Almost Periodic Fourier Series with time varying parameters and*** with limits $N \to \infty$, $M \to \infty$.

$$\Xi(x,t) = \sum_{n=-\infty}^{\infty} \Xi_n(\boldsymbol{\tau}(t)) e^{iK(t)x - i\Omega_n(t)t + i\Phi_n(t)}$$

[Osborne, 2010]

Step 3: The Nonintegrable Case 2+1 NLS+ Perturbations

Nonintegrable 2+1 NLS Plus Perturbations as Ratio of Product of Theta Functions, Quasiperiodic Fourier Series and Almost Periodic Fourier Series

Ansatz Solution For 2+1 NLS plus perturbations to *M* orders

$$\Xi(x,y,t) = a_o \frac{\prod_{m=1}^{M} \theta_m(x,y,t \mid \tau_m, \phi_m^-)}{\prod_{m=1}^{M} \theta_m(x,y,t \mid \tau_m, \phi_m^+)} e^{i[K(t)+k_o]x - i[\Omega(t)+\omega_o+\omega']t}$$

↓

Theta Functions With Slowly Varying Parameters to *M* orders

$$\theta_m(x,y,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} \theta_{m,\mathbf{n}}(\boldsymbol{\tau}_m(t)) e^{i\mathbf{n}\cdot\boldsymbol{\kappa}_m(t)x + i\mathbf{n}\cdot\boldsymbol{\Lambda}_m(t)y - i\mathbf{n}\cdot\boldsymbol{\omega}_m(t)t + i\mathbf{n}\cdot\boldsymbol{\phi}_m^{\mp}(t)}$$

$$\theta_{m,\mathbf{n}} = e^{\pi i \mathbf{n}\cdot\boldsymbol{\tau}_m(t)\mathbf{n}}$$

↓

Solution For 2+1 NLS plus perturbations to *M* orders as ***Quasiperiodic Fourier Series with Time Varying Parameters***: Single Valued, Multiply Periodic, Meromorphic Function with Slowly Varying Parameters

$$\Xi(x,y,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} \Xi_{\mathbf{n}}(t) e^{i\mathbf{n}\cdot\boldsymbol{\kappa}(t)x + i\mathbf{n}\cdot\boldsymbol{\Lambda}(t)y - i\mathbf{n}\cdot\boldsymbol{\omega}(t)t - i\mathbf{n}\cdot\boldsymbol{\Phi}(t)}$$

↓

Solution For 2+1 NLS plus perturbations to *M* Orders as ***Almost Periodic Fourier Series***

$$\Xi(x,y,t) = \sum_{n=-\infty}^{\infty} \Xi_n(t) e^{iK_n(t)x + i\Lambda_n(t)x - i\Omega_n(t)t + i\Phi_n(t)}$$

Determine the Nonlinear Fourier Spectrum in Terms of the Riemann Matrix and Phases I

Riemann Spectrum for 1+1 NLS from Surface Wave elevation

↓

Measured Space Series and Hilbert transform as function of time

$$\Xi(x,y,) = \eta(x,t) + i\tilde{\eta}(x,t)$$

↓

Estimate carrier amplitude, wavenumber, frequency and Stokes frequency correction from time series

$$a_o, k_o, \omega_o, \Omega'$$

↓

Compute Potential

$$U(x,t) = 2\left|\Xi(x,t)\right|^2$$

↓

Solve for Theta with Positive Phase

$$\partial_{xx} \ln F(x,t) = \frac{1}{2} U(x,t) = \left|\Xi(x,t)\right|^2$$

$$\theta(x,t \,|\, \boldsymbol{\tau}, \boldsymbol{\phi}^+) = F(x,t)$$

↓

1

Determine the Nonlinear Fourier Spectrum in Terms of the Riemann Matrix and Phases II

1

Compute Theta with Negative phase

$$H(x,t)=\Xi(x,t)F(x,t)\frac{e^{-ik_o x+i(\omega_o+\Omega')t}}{a_o}$$

$$\theta(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^{-})=H(x,t)$$

Compute Fourier Transform of Theta Functions

$$\theta(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^{\pm})=\sum_{n=-\infty}^{\infty}\theta_n^{\pm}(t)e^{ik_n x}$$

Compute Riemann Spectrum

$$\tau_{jj}=\frac{i}{2\pi}\ln\left[\left(\underbrace{\lim}_{T\to\infty}\frac{1}{T}\int_0^T\theta^{\pm}_{n=j}(t)e^{i\Omega_j t}dt\right)\left(\underbrace{\lim}_{T\to\infty}\frac{1}{T}\int_0^T\theta^{\pm}_{n=-j}(t)e^{-i\Omega_j t}dt\right)\right]$$

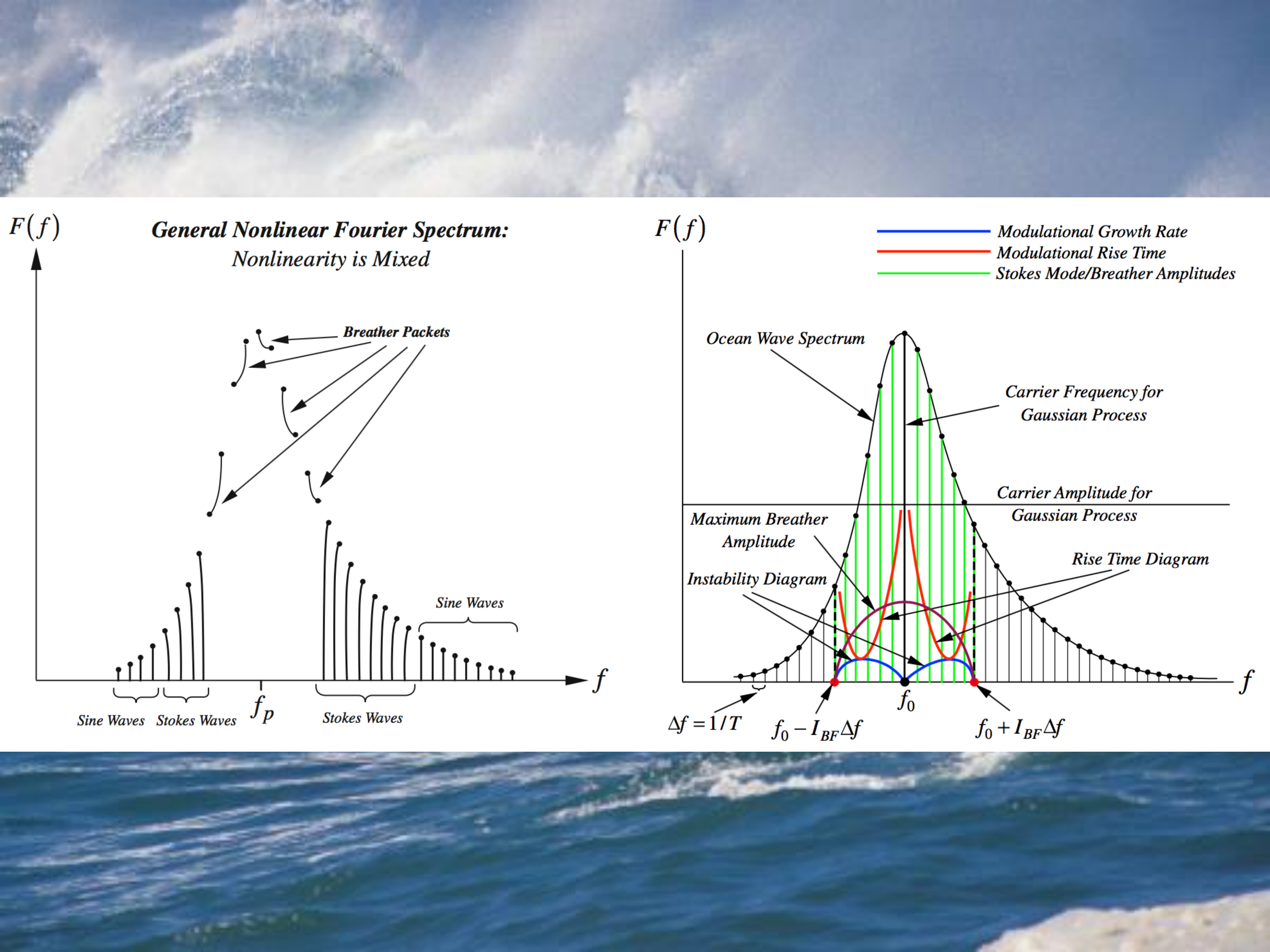
$$\tau_{jk}=\frac{i}{4\pi}\ln\left[\left(\underbrace{\lim}_{T\to\infty}\frac{1}{T}\int_0^T\theta^{\pm}_{n=j+k}(t)e^{i(\Omega_j+\Omega_k)t}dt\right)\left(\underbrace{\lim}_{T\to\infty}\frac{1}{T}\int_0^T\theta^{\pm}_{n=-(j+k)}(t)e^{-i(\Omega_j+\Omega_k)t}dt\right)\right]-\frac{1}{2}(\tau_{jj}+\tau_{kk})$$

$$\phi_j^{\pm}=-i\ln\left[\frac{\left(\underbrace{\lim}_{T\to\infty}\frac{1}{T}\int_0^T\theta^{\pm}_{n=j}(t)e^{i\mathbf{n}\cdot\boldsymbol{\Omega}t}dt\right)}{\left(\underbrace{\lim}_{T\to\infty}\frac{1}{T}\int_0^T\theta^{\pm}_{n=-j}(t)e^{-i\mathbf{n}\cdot\boldsymbol{\Omega}t}dt\right)}\right]$$

$\boldsymbol{\tau},\boldsymbol{\phi}^{\pm}$

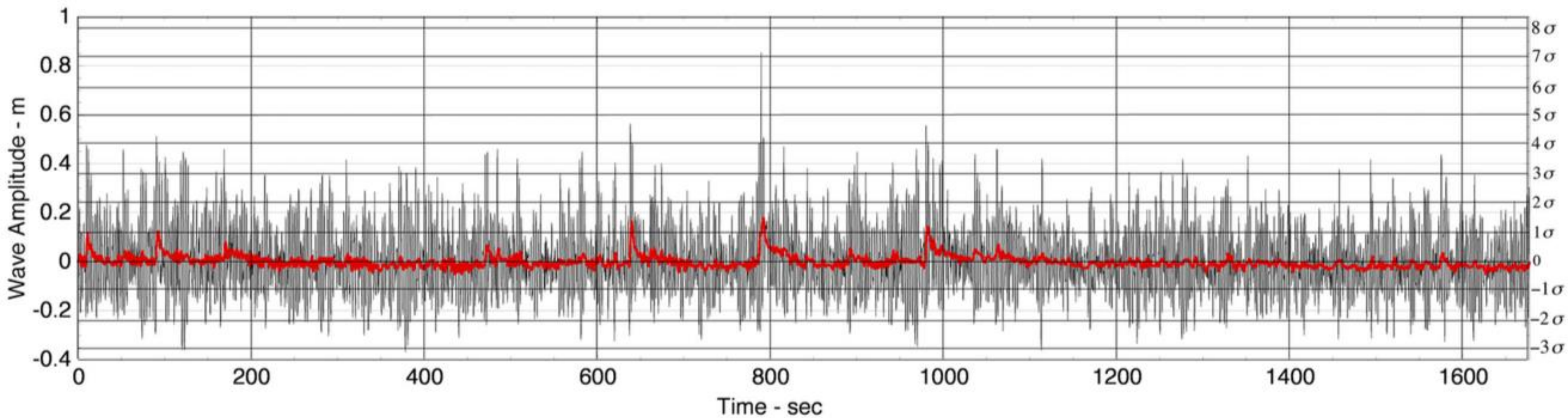
End

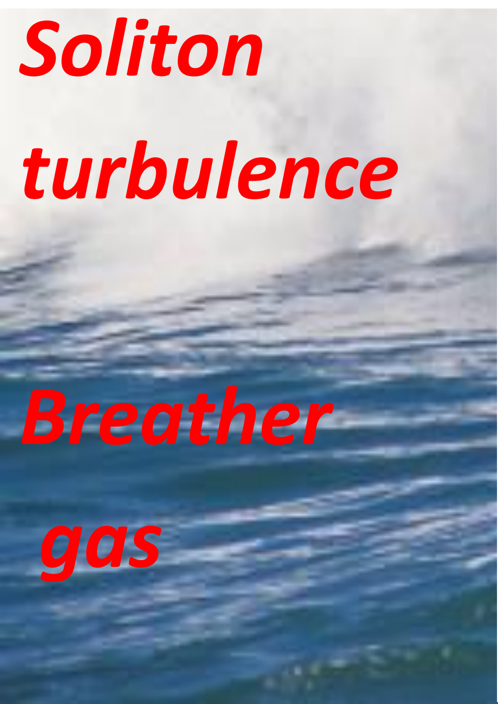
The Nonlinear Spectrum

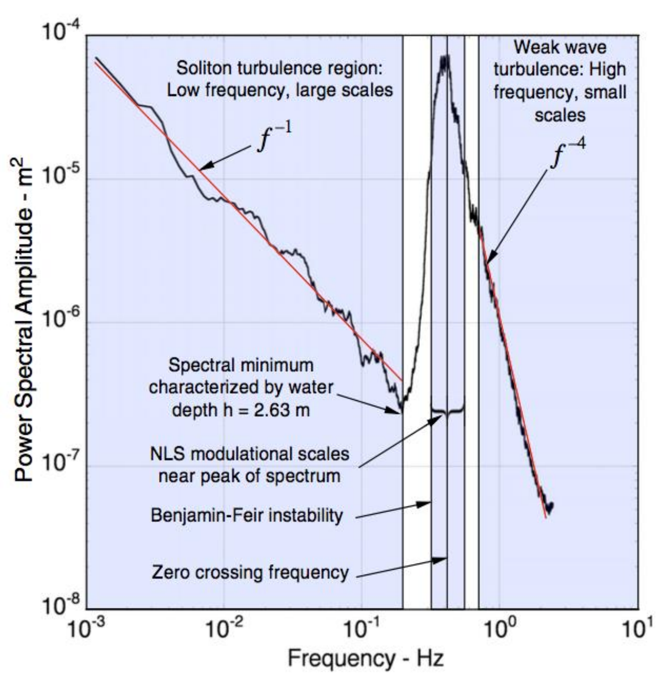

General Nonlinear Fourier Spectrum:
Nonlinearity is Mixed
F(f)
Breather Packets
Sine Waves
Sine Waves
Stokes Waves
Stokes Waves
f
fp
F(f)
Modulational Growth Rate
Modulational Rise Time
Stokes Mode/Breather Amplitudes
Ocean Wave Spectrum
Carrier Frequency for
Gaussian Process
Carrier Amplitude for
Gaussian Process
Maximum Breather
Amplitude
Instability Diagram
Rise Time Diagram
Δf = 1/T
f0 − IBF Δf
f0
f0 + IBF Δf
f

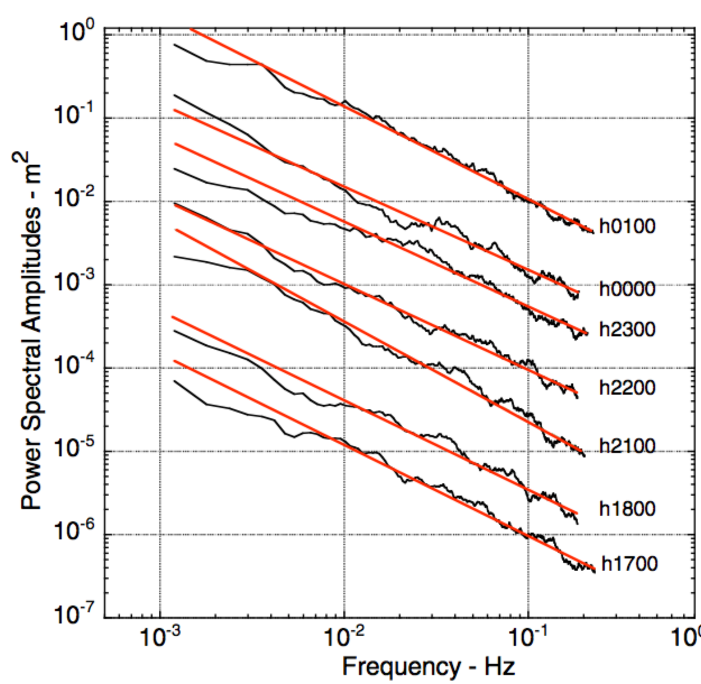

Currituck Sound

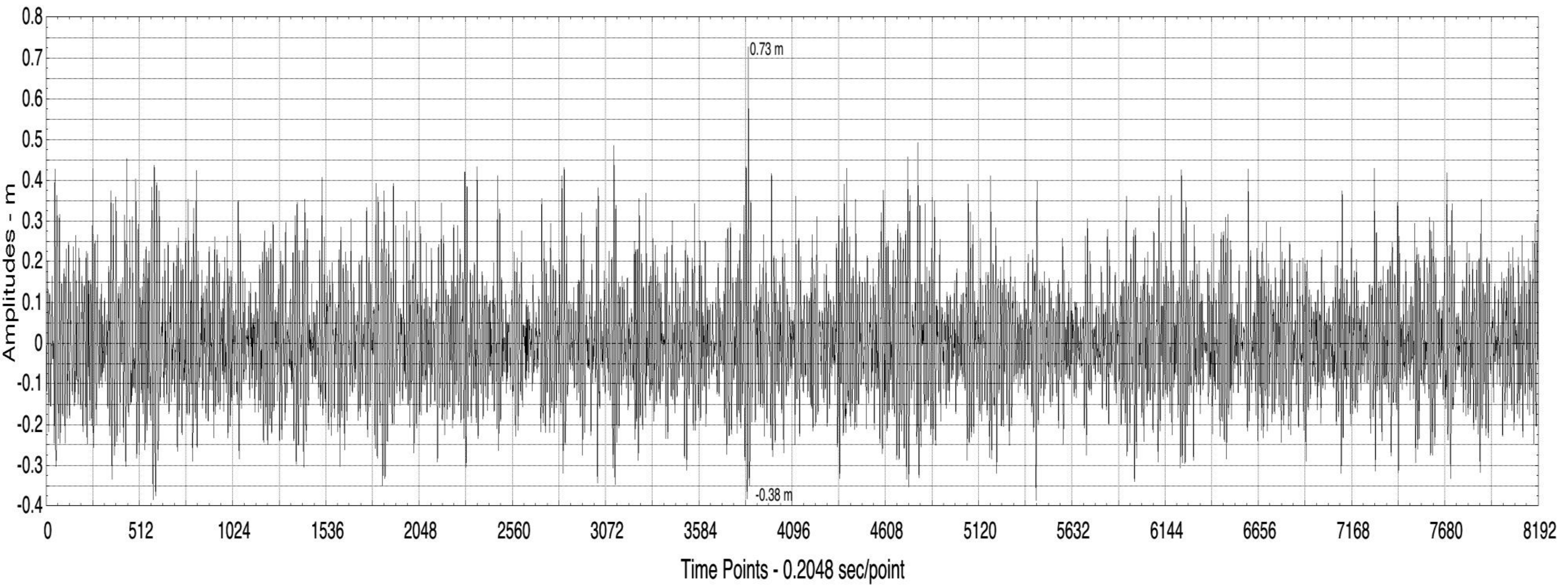
Understanding Nonlinear Data

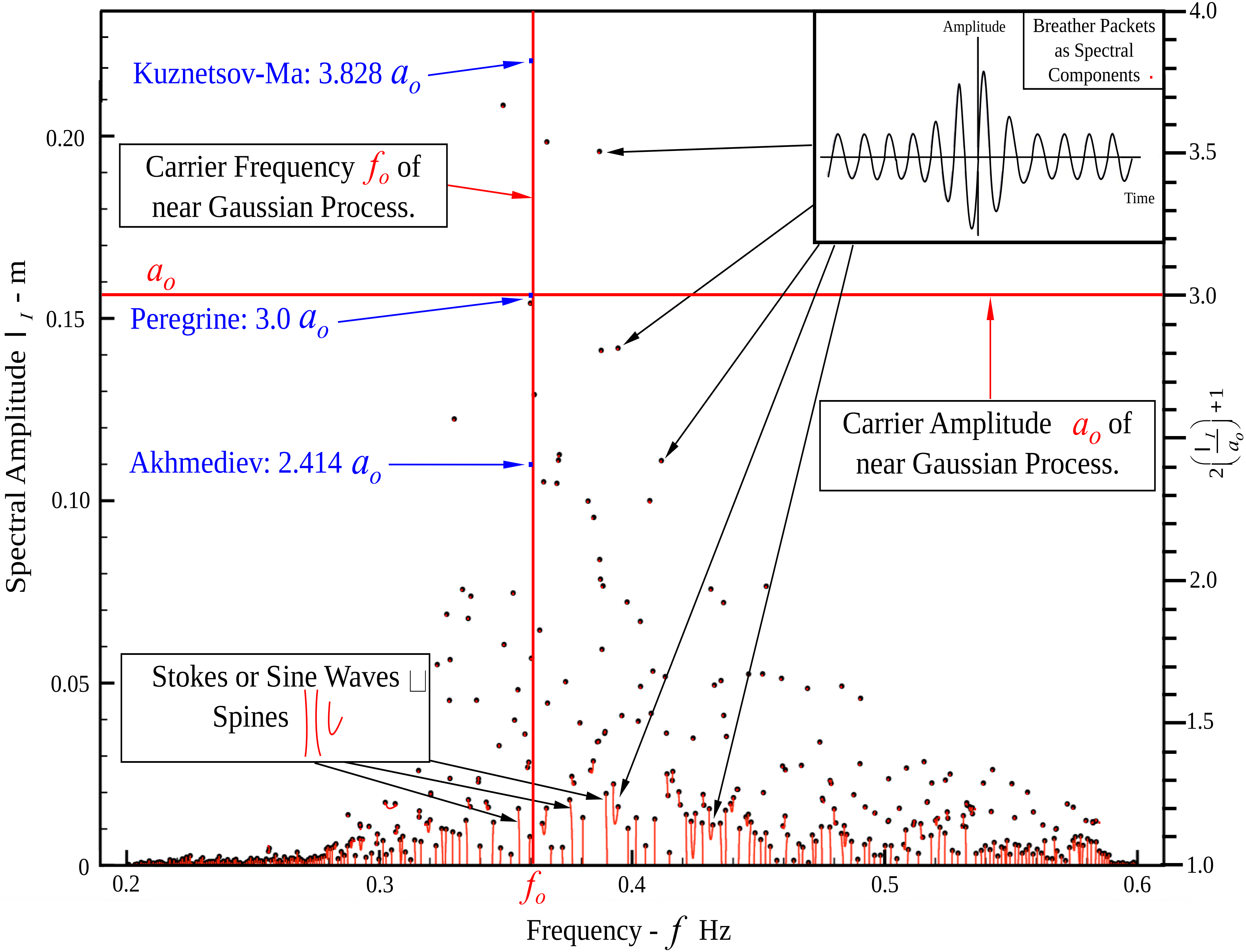









0.73 m
-0.38 m
Amplitudes - m
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
512
1024
1536
2048
2560
3072
3584
4096
4608
5120
5632
6144
6656
7168
7680
8192
Time Points - 0.2048 sec/point



Kuznetsov-Ma: 3.828 a_o
Carrier Frequency f_o of near Gaussian Process.
a_o
Peregrine: 3.0 a_o
Akhmediev: 2.414 a_o
Stokes or Sine Waves
Spines
Amplitude
Time
Breather Packets as Spectral Components
Carrier Amplitude a_o of near Gaussian Process.
Spectral Amplitude l_I - m
0.20
0.15
0.10
0.05
0
0.2
0.3
f_o
0.4
0.5
0.6
Frequency - f Hz
4.0
3.5
3.0
2.0
1.5
1.0
$2\left(\frac{l_I}{a_o}\right)+1$

Where the Breathers Are?

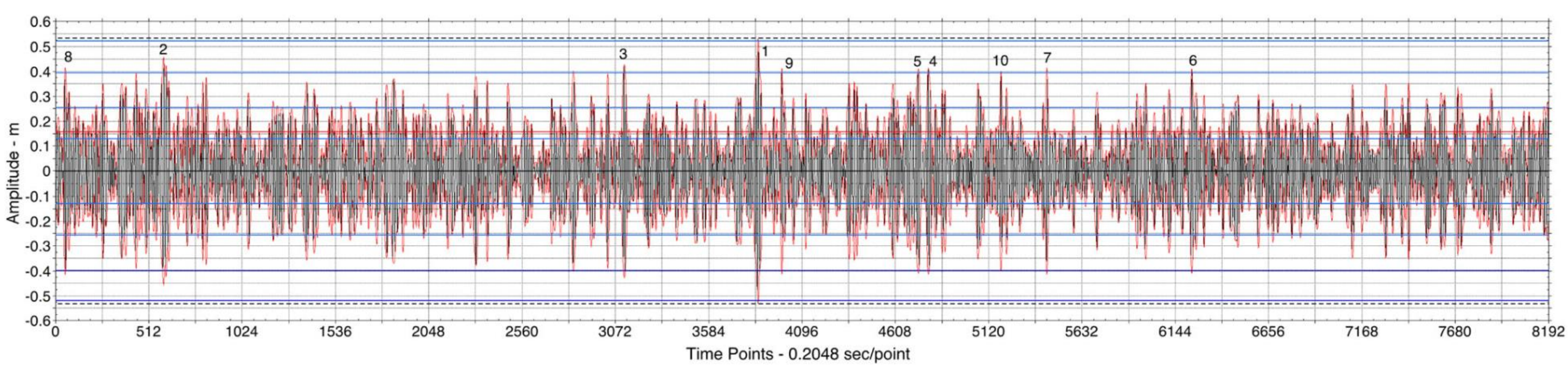


Table of Largest Breathers

No. Br.	NLFT Amp.	Max. Amp.	Max Height	Rise Distance /Rise Time
1	0.243 m	$4.06a_o$	$2.50H_s$	5.34 km/41.6 min
2	0.239 m	$3.99a_o$	$2.45H_s$	7.89 km/61.5 min
3	0.206 m	$3.58a_o$	$2.20H_s$	103. km/11.9 hrs
4	0.163 m	$3.05a_o$	$1.88H_s$	4.93 km/38.5 min
5	0.161 m	$3.02a_o$	$1.86H_s$	20.3 km/2.63 hrs
6	0.156 m	$2.95a_o$	$1.82H_s$	2.08 km/17.1 min
7	0.151 m	$2.90a_o$	$1.78H_s$	3.72 km/28.0 min
8	0.138 m	$2.74a_o$	$1.68H_s$	138. km/17.9 hrs